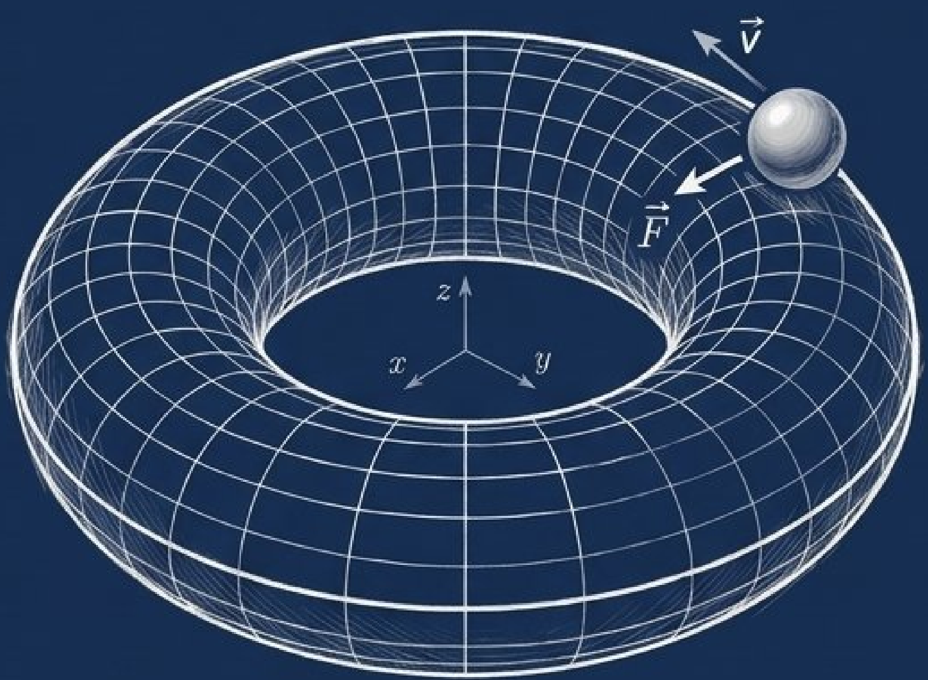
Classical Mechanics:
Principles of Newtonian Dynamics



Classical Mechanics: Principles of Newtonian Dynamics

A Calculus-Based Introduction

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Contents

Preface

The story of mechanics is, in many ways, the story of modern science itself. For two millennia, the dominant framework for understanding motion was that of Aristotle (384–322 BCE), who held that objects move only when a force is applied to them, that heavier objects fall faster than lighter ones, and that celestial bodies move in perfect circles because of their divine nature. These ideas were intuitively appealing and went essentially unchallenged for nearly two thousand years.

The revolution began with Galileo Galilei (1564–1642), who insisted that the laws of nature must be discovered by experiment, not by philosophical argument. His inclined-plane experiments demonstrated that all objects fall with the same acceleration regardless of mass, a direct contradiction of Aristotle. Equally important was his discovery of the *principle of inertia*: a body in motion on a frictionless surface continues moving indefinitely without any applied force. This idea, profoundly counterintuitive in a world full of friction, was the conceptual seed from which Newton's first law would grow.

Johannes Kepler (1571–1630), working from the meticulous observational data of Tycho Brahe, distilled the motion of the planets into three elegant empirical laws: elliptical orbits, equal areas in equal times, and the precise relationship $T^2 \propto a^3$ between orbital period and semi-major axis. Kepler's laws were descriptive: they said *what* the planets do, but not *why*.

It fell to Isaac Newton (1643–1727) to provide the *why*. In his *Philosophiæ Naturalis Principia Mathematica* (1687), Newton achieved one of the greatest intellectual syntheses in history. He formulated three laws of motion and the law of universal gravitation, then showed that Kepler's empirical laws follow as mathematical consequences. The same force that causes an apple to fall from a tree holds the Moon in orbit around the Earth and the Earth in orbit around the Sun. This unification of terrestrial and celestial mechanics was revolutionary: for the first time, the heavens and the Earth were governed by the same physics.

Newton also co-invented the calculus (independently of Leibniz) precisely because the mathematics of his era was insufficient for the problems he was trying to solve. The kinematic equations, the work-energy theorem, the derivation of orbital trajectories: all require derivatives and integrals. This is why calculus appears throughout this textbook: not as an optional mathematical garnish, but as the language in which the laws of mechanics are naturally expressed.

The century after Newton saw the reformulation and extension of mechanics by Euler, Lagrange, Hamilton, and others. Leonhard Euler (1707–1783) developed the theory of rigid-body rotation and the calculus of variations. Joseph-Louis Lagrange (1736–1813) recast all of Newtonian mechanics in terms of energy functions and generalized coordinates, producing a formulation that is more powerful and more elegant than Newton's original vector approach. William Rowan Hamilton (1805–1865) went further still, reformulating mechanics in terms of phase space and canonical transformations—ideas that would prove essential to the development of quantum mechanics a century later. These "analytical mechanics" formulations are the subject of more advanced courses; the Newtonian approach developed in this book is the essential foundation on which they rest.

Classical mechanics is extraordinarily successful within its domain. It accurately describes the motion of objects ranging in size from dust grains to galaxies, moving at speeds well below the speed of light, and subject to gravitational fields that are not too extreme. When these conditions are violated, classical mechanics must be replaced by more general theories: *special relativity* (Einstein, 1905) for speeds approaching c, *general relativity* (Einstein, 1915) for strong gravitational fields and the curvature of spacetime, and *quantum mechanics* (Planck, Bohr, Heisenberg, Schrödinger, Dirac, 1900–1930) for atomic and subatomic phenomena. Remarkably, each of these theories reduces to Newtonian mechanics in the appropriate limit, a deep consistency requirement that any valid physical theory must satisfy.

Despite being over three centuries old, Newtonian mechanics remains indispensable in science and engineering. Every trajectory computed by NASA—from the Apollo Moon landings to the Mars rovers to the Voyager spacecraft now leaving the solar system—is calculated using the equations developed in this book (with small relativistic corrections where needed). The design of bridges, buildings, vehicles, and machines relies on the principles of statics, dynamics, and rotational equilibrium covered in our chapters on Newton's laws, energy, and rigid-body mechanics. Biomechanics applies these same principles to the human body: the torques in a knee joint, the forces on a spinal column, the energy efficiency of locomotion.

In modern physics research, classical mechanics forms the starting point for virtually every subfield. Chaos theory and nonlinear dynamics (active areas of current research) are built directly on the Newtonian framework. Astrophysicists use Kepler's laws and gravitational dynamics to model everything from exoplanetary systems to the large-scale structure of the universe. Even in quantum mechanics, the classical limit provides essential physical intuition: the correspondence principle asserts that quantum predictions must agree with classical ones for large quantum numbers, and the path-integral formulation of quantum mechanics is built on the classical action.

The goal of this textbook is to give you a rigorous, physically motivated understanding and an appreciation of this foundational subject: one that will serve you whether you continue into advanced physics, engineering, or any field where quantitative reasoning about the physical world is required.

This textbook presents the foundations of classical mechanics at the introductory university level, written for students with a working knowledge of single-variable calculus and some exposure to multivariable calculus. The treatment is calculus-based throughout: derivatives, integrals, and differential equations appear wherever the physics demands them, because the language of calculus is the natural language of mechanics.

The text is organized into three main parts. **Part I** develops the concepts in a sequence of chapters, each building on the last: from the nature of physical measurement, through vectors, kinematics, Newton's laws, energy, momentum, rotational dynamics, gravitation, and oscillatory motion. Every physics theorem is proved where it is first stated—the rocket equation is derived in the momentum chapter, the shell theorem is proved in the gravitation chapter, and so on—so that the reader sees the physical reasoning and mathematical technique together. **Part II** collects the solutions to every problem posed in Part I. **Part III**, the Appendix, is a purely mathematical reference: calculus review, Taylor series, trigonometric identities, solutions to ordinary differential equations, multivariable calculus, and linear algebra—tools that are deployed throughout the main text but whose detailed development would interrupt the flow of the physics.

The problems, inspired by many classic sources, range in difficulty from single-step exercises ($\star$) to multi-part problems requiring synthesis of several ideas ($\star\star\star\star\star$).

A few philosophical notes on the presentation:

- *Self-contained problem statements.* Every problem states the principles and data needed for its solution; students should never have to make unstated assumptions.
- *Physical motivation first.* Definitions and theorems are motivated by physical questions before they are stated formally.
- *Calculus as a tool, not an obstacle.* The mathematical machinery is developed in the Appendix and deployed in the main text as needed, always in service of understanding the physics.

Part I

Concepts

Chapter 1

The Nature of Physics

1.1 What is Physics?

Physics is the study of the fundamental principles governing the natural world. Physics, intrinisically, seeks to describe: *how* things move, *why* they move, and *what* constrains or enables that motion. The scope of classical mechanics (the subject of this book) spans everything from the trajectory of a thrown ball to the orbits of planets around the Sun.

The power of physics lies in its ability to express the laws of nature in precise mathematical language. Newton's second law, $\boldsymbol{F}_{\text{net}} = m\boldsymbol{a}$, is not merely a formula to be memorized; it is a compact statement that encodes the relationship between force and motion for *every* macroscopic object in the universe (at speeds well below the speed of light and at scales well above the atomic).

Models, Theories, and Laws

Physics proceeds by constructing *models*—simplified representations of physical systems that capture the essential features while ignoring irrelevant complications. A block sliding on a surface is modeled as a point particle; a planet is modeled as a sphere of uniform density; a spring is modeled as a massless, perfectly elastic element. These idealizations are not defects but deliberate choices that make the mathematics tractable. The art of physics lies in knowing which features to keep and which to discard.

A *theory* is a broader explanatory framework that unifies many observations and predictions. Newton's theory of gravitation, for example, explains not only the motion of planets but also the tides, the precession of the equinoxes, and the trajectories of comets: all from the law of universal gravitation combined with the three laws of motion. A theory is never "proven" in the mathematical sense; it is tested against experiment, and it stands only as long as it continues to agree with observation.

A *law* is a concise mathematical statement of a pattern observed in nature. Newton's three laws of motion and his law of universal gravitation are examples. Laws are descriptive (they say *what* happens) rather than explanatory (they do not say *why*). The distinction between a law and a theory is sometimes blurred, but the key point is that all of physics rests on empirical evidence: no amount of mathematical elegance can substitute for agreement with experiment.

Solving Physics Problems

Throughout this textbook, you will encounter problems of increasing complexity. A systematic approach will serve you far better than memorized formulas. We recommend the following four-step framework.

Strategy 1.1: Problem-Solving Strategy

1. **Identify.** Read the problem carefully. What physical principles are involved? What are you asked to find? Draw a diagram if possible.
2. **Set up.** Choose a coordinate system. List the known and unknown quantities. Identify the equations that connect them. This is the step where most of the physics happens.
3. **Execute.** Solve the equations. Work algebraically as long as possible before substituting numbers; this reduces errors and reveals how the answer depends on the given quantities.
4. **Evaluate.** Check your answer. Is it dimensionally correct? Does the sign make sense? Does it reduce to a known result in a special case (e.g., $m \to 0$ or $\theta \to 0$)? Is the numerical value reasonable?

The "evaluate" step is especially important and is the one most often skipped. If you derive that a car's acceleration is $500\,\mathrm{m/s^2}$, something has gone wrong: that is roughly 50 times the acceleration of gravity. Dimensional checks catch algebraic errors; limiting-case checks catch conceptual errors; order-of-magnitude checks catch arithmetic errors. Cultivating the habit of checking your answers will save you more points on exams than any formula sheet ever could.

1.2 Physical Quantities and the SI System

Every measurement in physics consists of a *number* and a *unit*. A statement like "the length is 5" is meaningless; "the length is 5 m" conveys information.

Definition 1.1: The International System of Units (SI)

The SI system defines seven base quantities from which all others are derived:

Quantity	**Unit**	**Symbol**
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

In mechanics, we primarily use the first three: meters, kilograms, and seconds.

Derived units are constructed from the base units. For example, the unit of force is the *newton*: $1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}$. The unit of energy is the *joule*: $1\,\mathrm{J} = 1\,\mathrm{N\,m} = 1\,\mathrm{kg\,m^2/s^2}$. The unit of power is the *watt*: $1\,\mathrm{W} = 1\,\mathrm{J/s}$.

SI Prefixes

It is often convenient to express very large or very small quantities using SI prefixes. The most common in mechanics are:

Prefix	**Symbol**	**Factor**	**Example**
giga	G	10^9	$1\,\text{GJ} = 10^9$ J
mega	M	10^6	$1\,\text{MW} = 10^6$ W
kilo	k	10^3	$1\,\text{km} = 10^3$ m
centi	c	10^{-2}	$1\,\text{cm} = 10^{-2}$ m
milli	m	10^{-3}	$1\,\text{mm} = 10^{-3}$ m
micro	μ	10^{-6}	$1\,\mu\text{m} = 10^{-6}$ m
nano	n	10^{-9}	$1\,\text{nm} = 10^{-9}$ m

1.2.1 The Scales of Nature

One of the most remarkable features of physics is the sheer range of scales it encompasses. The following tables give a sense of the orders of magnitude encountered in the physical world.

Length (m)	**Object**
10^{-15}	Diameter of a proton
10^{-10}	Diameter of an atom
10^{-8}	Diameter of a virus
10^{-5}	Diameter of a red blood cell
10^{0}	Human height
10^{4}	Diameter of a city
10^{7}	Diameter of the Earth
10^{11}	Earth–Sun distance (1 AU)
10^{16}	Distance to nearest star (Proxima Centauri)
10^{21}	Diameter of the Milky Way galaxy
10^{26}	Radius of the observable universe

Mass (kg)	**Object**
10^{-30}	Electron
10^{-27}	Proton
10^{-15}	Bacterium
10^{-6}	Mosquito
10^{-3}	Hummingbird
10^{2}	Human
10^{3}	Car
10^{24}	Earth
10^{30}	Sun
10^{42}	Milky Way galaxy
10^{53}	Observable universe

Time (s)	**Duration**
10^{-23}	Time for light to cross a proton
10^{-15}	Period of visible light
10^{-3}	Camera flash
10^{0}	Human heartbeat
10^{5}	One day (8.64×10^4 s)
10^{7}	One year (3.15×10^7 s)
10^{11}	Recorded human history (∼5000 yr)
10^{17}	Age of the universe (4.3×10^{17} s)

These tables illustrate that the quantities studied in physics span roughly 40 orders of magnitude in length, over 80 in mass, and 40 in time. Classical mechanics applies over the vast middle range of these scales—from roughly 10^{-4} m to 10^{26} m in length, from dust grains to galaxy clusters in mass, and from milliseconds to billions of years in time. Only at the extremes—the very small (quantum mechanics) and the very fast or strongly gravitating (relativity)—do we need to go beyond Newton.

1.2.2 The Standards of Length, Time, and Mass

A unit is only as good as the *standard* that defines it. For a standard to be useful, it must be both *accessible* (reproducible in any well-equipped laboratory) and *invariable* (not subject to change over time). The history of the SI base units is a story of progressively replacing human-made artifacts with definitions based on fundamental constants of nature.

Length. The meter was originally defined in 1791 by the French Academy of Sciences as one ten-millionth of the distance from the North Pole to the equator along the meridian through Paris. This was replaced in 1889 by the distance between two engraved lines on a platinum-iridium bar kept at the International Bureau of Weights and Measures (BIPM) near Paris. In 1960, the definition shifted to 1,650,763.73 wavelengths of the orange-red light emitted by krypton-86 atoms, a standard based on atomic physics rather than a human artifact. Since 1983, the meter has been defined as the distance traveled by light in vacuum in exactly 1/299,792,458 of a second. This definition *fixes* the speed of light to be exactly $c = 299\,792\,458\,\mathrm{m/s}$.

Time. The second was historically defined as 1/86,400 of a mean solar day. However, the Earth's rotation is not perfectly uniform: it is gradually slowing due to tidal friction and exhibits irregular fluctuations. Since 1967, the second has been defined as the duration of 9,192,631,770 oscillations of the radiation emitted during a specific transition of the cesium-133 atom. Modern cesium atomic clocks are accurate to about 1 second in 300 million years, and next-generation optical lattice clocks achieve even higher precision.

Mass. The kilogram has the most dramatic history. From 1889 until 2019, it was defined as the mass of a single physical object: the International Prototype of the Kilogram (IPK), a platinum-iridium cylinder stored under three nested bell jars in a vault near Paris. This was the last SI unit defined by a human-made artifact, and it posed a fundamental problem: comparisons between the IPK and its copies showed that their relative masses were drifting by tens of micrograms per century, and there was no way to determine which one (if any) was "correct."

In 2019, the kilogram was redefined in terms of the Planck constant, $h = 6.626\,070\,15 \times 10^{-34}\,\mathrm{J\,s}$ (exact, by definition). Since the joule involves $\mathrm{kg \cdot m^2/s^2}$ and the meter and second are already defined via c and $\Delta\nu_{\mathrm{Cs}}$, fixing h implicitly fixes the kilogram. The practical realization uses a device called a *Kibble balance*, which compares mechanical and electrical power to relate mass to h with extraordinary precision. As of the 2019 redefinition, *all* SI base units are defined in terms of fundamental constants of nature: no physical artifacts remain.

1.2.3 Density

One of the most important derived quantities in mechanics is *density*: the mass per unit volume.

$$\rho = \frac{m}{V}. \tag{1.1}$$

The SI unit of density is $\mathrm{kg/m^3}$. Some representative densities: air at sea level $\approx 1.2\,\mathrm{kg/m^3}$; water $= 1000\,\mathrm{kg/m^3}$; iron $\approx 7870\,\mathrm{kg/m^3}$; the Sun $\approx 1410\,\mathrm{kg/m^3}$; a neutron star $\sim 10^{17}$ $\mathrm{kg/m^3}$. The enormous range (spanning more than 20 orders of magnitude) reflects the diversity of physical systems that mechanics must describe.

For uniform objects, ρ is constant throughout the body, and the total mass is simply $m = \rho V$. For non-uniform objects, we must integrate: $m = \int \rho\, dV$. We will encounter this integral frequently when computing centers of mass and moments of inertia in later chapters.

1.3 Dimensional Analysis

Every physical equation must be *dimensionally consistent*: the dimensions on each side of the equation must match. Dimensional analysis is a powerful tool for checking equations, deriving relations, and estimating unknown quantities.

Definition 1.2: Dimensions

The *dimension* of a quantity specifies its type (length, mass, time, etc.) without reference to a particular unit system. We write dimensions using square brackets:

$$[x] = \mathsf{L}, \qquad [m] = \mathsf{M}, \qquad [t] = \mathsf{T}.$$

The dimensions of any mechanical quantity can be expressed as $\mathsf{M}^a\mathsf{L}^b\mathsf{T}^c$ for some exponents a, b, c.

For example, velocity has dimensions $[v] = \mathsf{LT}^{-1}$, acceleration has $[a] = \mathsf{LT}^{-2}$, and force has $[F] = \mathsf{MLT}^{-2}$. Energy has $[E] = \mathsf{ML}^2\mathsf{T}^{-2}$, and pressure has $[P] = [F/A] = \mathsf{ML}^{-1}\mathsf{T}^{-2}$.

The requirement of dimensional consistency places strong constraints on the form of physical laws. If someone proposes that the period of a pendulum is $T = 2\pi m\ell/g$, you can immediately reject this: $[m\ell/g] = \mathsf{M} \cdot \mathsf{L}/(\mathsf{LT}^{-2}) = \mathsf{MT}^2$, which has dimensions of mass $\times$ time^2, not time. No further physics is needed to identify the error.

Theorem 1.1: The Buckingham Pi Theorem (Informal Statement)

If a physical law involves n dimensional quantities and k independent base dimensions, then the law can be expressed in terms of $n - k$ independent dimensionless combinations (called Π-groups).

This theorem provides a systematic framework for dimensional analysis. In the simplest case, where $n - k = 1$, the single dimensionless group must equal a constant, and dimensional analysis determines the functional form of the law completely (up to that dimensionless constant).

Example 1.1 (Period of a simple pendulum). Suppose the period T of a simple pendulum depends only on its length ℓ, the mass m of the bob, and the gravitational acceleration g. Then $T = C\, m^a \ell^b g^c$ for some dimensionless constant C. Matching dimensions:

$$\mathsf{T} = \mathsf{M}^a\mathsf{L}^b(\mathsf{LT}^{-2})^c = \mathsf{M}^a\mathsf{L}^{b+c}\mathsf{T}^{-2c}.$$

This gives $a = 0$, $-2c = 1$ so $c = -1/2$, and $b + c = 0$ so $b = 1/2$. Therefore

$$T = C\sqrt{\frac{\ell}{g}},$$

and dimensional analysis alone determines the functional form up to the dimensionless constant C (which turns out to be 2π). Notice that m drops out entirely: the period of a pendulum is independent of the mass of the bob. This is a nontrivial physical prediction obtained from dimensional reasoning alone.

Common Mistake 1.1: Dimensional Traps

Dimensional consistency is necessary but not sufficient. The equation $x = v_0 t + at^2$ is dimensionally correct but physically wrong (the correct equation has $\frac{1}{2}at^2$). Dimensional analysis checks the *type* of each term; it cannot determine numerical coefficients. Furthermore, the arguments of transcendental functions (sin, cos, exp, ln) must always be dimensionless. An expression like $\sin(5\,\mathrm{m})$ is meaningless.

Estimation and Fermi Problems

Physicists frequently estimate quantities to within an order of magnitude. Such estimates—called *Fermi problems* after Enrico Fermi, who was famous for his ability to produce quick, accurate estimates from minimal data—develop physical intuition and the ability to identify the dominant factors in a complex situation.

The key strategy is to break the problem into a chain of simpler estimates, express each factor as a power of 10, and multiply. Errors in individual factors tend to cancel (some overestimate, some underestimate), so the final result is typically reliable to within a factor of 3–10.

Example 1.2 (How many golf balls fit in a school bus?) A school bus interior is roughly $2.5\,\mathrm{m} \times 2\,\mathrm{m} \times 10\,\mathrm{m} = 50\,\mathrm{m}^3$. A golf ball has diameter $\sim 4\,\mathrm{cm}$, so volume $\sim \frac{4}{3}\pi(0.02)^3 \approx 3.4 \times 10^{-5}\,\mathrm{m}^3$. Random packing fills about 64% of space. Number $\approx 0.64 \times 50/3.4 \times 10^{-5} \approx 940{,}000$. Order of magnitude: $\sim 10^6$.

Significant Figures and Uncertainty

Every experimental measurement carries uncertainty. If you measure a table's length with a meter stick and find $\ell = 1.52\,\mathrm{m}$, you are implicitly claiming that the true length lies somewhere near $1.52\,\mathrm{m}$—probably between $1.515\,\mathrm{m}$ and $1.525\,\mathrm{m}$. The number of *significant figures* communicates this precision: $1.52\,\mathrm{m}$ has three significant figures, meaning the measurement is reliable to parts in 10^3.

Rules for significant figures. When multiplying or dividing, the result should have the same number of significant figures as the least precise input. When adding or subtracting, the result should be rounded to the same decimal place as the least precise input. Constants like π or exact conversion factors do not limit significant figures.

More formally, every measurement can be written as $x \pm \delta x$, where δx is the *absolute uncertainty*. The *fractional* (or *relative*) *uncertainty* is the ratio $\delta x/x$, often expressed as a percentage. For example, $\ell = 1.52\,\mathrm{m} \pm 0.005\,\mathrm{m}$ has a fractional uncertainty of $0.005/1.52 \approx 0.3\%$.

Key Point 1.1: Propagation of Uncertainty

When quantities with uncertainties are combined:

- **Addition/subtraction**: $q = a + b$ gives $\delta q = \delta a + \delta b$ (absolute uncertainties add).
- **Multiplication/division**: $q = ab$ gives $\delta q/|q| = \delta a/|a| + \delta b/|b|$ (fractional uncertainties add).
- **Powers**: $q = a^n$ gives $\delta q/|q| = |n| \cdot \delta a/|a|$.

These are "worst-case" rules. In advanced treatments, uncertainties are added in quadrature (root-sum-of-squares) assuming independent random errors, but the linear rules above are standard for introductory physics.

Example 1.4 (Uncertainty in computed density). A cylinder has mass $m = 5.42\,\text{g} \pm 0.03\,\text{g}$, diameter $d = 1.20\,\text{cm} \pm 0.02\,\text{cm}$, and height $h = 3.15\,\text{cm} \pm 0.02\,\text{cm}$. Find the density and its uncertainty.

Volume: $V = \pi(d/2)^2 h = \pi(0.60)^2(3.15) = 3.563\,\text{cm}^3$. Density: $\rho = m/V = 5.42/3.563 = 1.52\,\text{g/cm}^3$.

Fractional uncertainties: $\delta m/m = 0.03/5.42 = 0.55\%$; $\delta d/d = 0.02/1.20 = 1.67\%$ (enters squared since $V \propto d^2$, so contributes $2 \times 1.67\% = 3.33\%$); $\delta h/h = 0.02/3.15 = 0.63\%$. Total fractional uncertainty in ρ: $0.55\% + 3.33\% + 0.63\% = 4.5\%$. So $\delta\rho = 0.045 \times 1.52 = 0.07\,\text{g/cm}^3$, and we report $\rho = 1.52\,\text{g/cm}^3 \pm 0.07\,\text{g/cm}^3$. Note that the diameter measurement dominates the uncertainty because it enters squared.

In this textbook, we generally use $g = 9.8\,\text{m/s}^2$ (or $g = 10\,\text{m/s}^2$ when stated in the problem) and work to three significant figures unless otherwise noted.

Unit Conversion

Converting between units is accomplished by multiplying by conversion factors equal to 1. For example:

$$60\,\text{mph} = 60\,\frac{\text{mi}}{\text{hr}} \times \frac{1609\text{ m}}{1\text{ mi}} \times \frac{1\text{ hr}}{3600\text{ s}} = 26.8\,\text{m/s}.$$

A useful shortcut: $1\,\text{m/s} \approx 2.24\,\text{mph}$ and $1\,\text{m/s} \approx 3.6\,\text{km/hr}$.

When converting areas and volumes, remember that the conversion factor must be raised to the appropriate power. For instance, $1\,\text{m}^2 = 10^4\,\text{cm}^2$ (not $100\,\text{cm}^2$), because $1\,\text{m} = 100\,\text{cm}$ implies $1\,\text{m}^2 = (100)^2\,\text{cm}^2$. Similarly, $1\,\text{m}^3 = 10^6\,\text{cm}^3$, so a density of $1\,\text{g/cm}^3 = 1000\,\text{kg/m}^3$.

Example 1.5 (Mass of the Earth's atmosphere). Estimate the total mass of Earth's atmosphere.

The atmosphere exerts a pressure of approximately $1.01 \times 10^5\,\text{Pa}$ (1 atm) at sea level. Pressure is force per unit area, so the total weight of the atmosphere is $W = P \cdot A$, where $A = 4\pi R_E^2$ is Earth's surface area. With $R_E \approx 6.4 \times 10^6\,\text{m}$:

$$A = 4\pi(6.4 \times 10^6)^2 \approx 5.1 \times 10^{14}\,\text{m}^2.$$

The total weight is $W = (1.01 \times 10^5)(5.1 \times 10^{14}) \approx 5.2 \times 10^{19}\,\text{N}$, so $m = W/g \approx 5.2 \times 10^{19}/9.8 \approx 5.3 \times 10^{18}\,\text{kg}$. This is about one millionth of Earth's mass ($6.0 \times 10^{24}\,\text{kg}$)—a reassuringly thin shell of gas.

Worked Examples

Example 1.6 (Period of a satellite). A satellite orbits a planet of mass M at radius r. Find the period T using dimensional analysis.

The relevant variables are T, M, r, and G (since gravity provides the force). Their dimensions:

$$[T] = \mathsf{T}, \quad [M] = \mathsf{M}, \quad [r] = \mathsf{L}, \quad [G] = \mathsf{M}^{-1}\mathsf{L}^3\mathsf{T}^{-2}.$$

Assume $T = C\,G^a M^b r^c$:

$$\mathsf{T} = (\mathsf{M}^{-1}\mathsf{L}^3\mathsf{T}^{-2})^a \mathsf{M}^b \mathsf{L}^c = \mathsf{M}^{-a+b}\mathsf{L}^{3a+c}\mathsf{T}^{-2a}.$$

From T: $-2a = 1$, so $a = -1/2$. From M: $b = a = -1/2$. From L: $c = -3a = 3/2$.

$$T = C\,\frac{r^{3/2}}{\sqrt{GM}} = C\sqrt{\frac{r^3}{GM}}.$$

The exact result (from Newton's laws) gives $C = 2\pi$. Dimensional analysis determined the functional dependence entirely; only the pure number 2π was left undetermined. Note that this result immediately implies Kepler's third law: $T^2 \propto r^3$.

Problems

Problem 1.1 ⋆

What are the dimensions of (a) energy, (b) power, (c) pressure, (d) the gravitational constant G, and (e) the spring constant k?

Problem 1.2 ⋆⋆

The drag force on a sphere moving through a fluid depends on the sphere's radius R, its speed v, and the fluid density ρ. Use dimensional analysis to determine how F_{drag} depends on these quantities.

Problem 1.3 ⋆⋆

A student writes $v^2 = v_0^2 + 2ax^3$. Without doing any physics, explain why this equation must be wrong.

Problem 1.4 ⋆ ⋆ ⋆

The speed of waves on a stretched string depends on the tension T (units of force) and the linear mass density μ (units of kg/m). Use dimensional analysis to find the wave speed.

Problem 1.5 ⋆ ⋆ ⋆

Estimate the number of piano tuners in a city of 3 million people. State all your assumptions clearly.

Problem 1.6 ⋆⋆

The period of a mass on a spring is $T = 2\pi\sqrt{m/k}$. Verify this is dimensionally correct.

Problem 1.7 ⋆ ⋆ ⋆

The centripetal acceleration of a planet in circular orbit depends on the orbital radius r and period T. Use dimensional analysis to find $a_c(r, T)$.

Problem 1.8 ⋆⋆

Convert: (a) $100\,\text{km/hr}$ to m/s, (b) $1\,\text{year}$ to seconds, (c) $1\,\text{light} - \text{year}$ to meters.

Problem 1.9 ⋆ ⋆ ⋆

The gravitational force on a planet near a star depends on the star's luminosity L (W), the speed of light c (m/s), and distance r (m). Using dimensional analysis, find how the radiation force depends on these quantities.

Problem 1.10 ⋆ ⋆ ⋆⋆

Estimate the total kinetic energy of all the air molecules in this room. State your assumptions about room size, air density, temperature, and molecular speed.

Chapter 2

Vectors and Coordinate Systems

Much of physics involves quantities that have both magnitude and direction—velocity, force, acceleration, momentum. To work with these quantities precisely, we need the language of **vectors** and the machinery of **coordinate systems**. This chapter develops those tools, which will be used in every subsequent chapter of this book.

2.1 Scalars and Vectors

Definition 2.1: Scalar and Vector

A **scalar** is a physical quantity completely specified by a number with units. Examples include mass, time, temperature, energy, and speed.
A **vector** is a physical quantity that requires both a magnitude (a non-negative scalar) and a direction. Examples include displacement, velocity, acceleration, force, and momentum.

We denote vectors in boldface: $\boldsymbol{A}$. The magnitude of $\boldsymbol{A}$ is written $|\boldsymbol{A}|$ or simply A. Two vectors are equal if and only if they have the same magnitude *and* the same direction. This means a vector can be freely translated (moved parallel to itself) without changing its identity; only its magnitude and direction matter, not where it is drawn.

The *zero vector* $\mathbf{0}$ has magnitude zero and no defined direction. It is the additive identity: $\boldsymbol{A} + \mathbf{0} = \boldsymbol{A}$ for any vector $\boldsymbol{A}$.

Unit Vectors

A **unit vector** is a vector of magnitude 1 that specifies a direction only. Given any non-zero vector $\boldsymbol{A}$, the corresponding unit vector is

$$\hat{\boldsymbol{A}} = \frac{\boldsymbol{A}}{|\boldsymbol{A}|}. \tag{2.1}$$

The hat notation $\hat{\boldsymbol{A}}$ always denotes a unit vector. We can reconstruct $\boldsymbol{A}$ from its magnitude and direction as $\boldsymbol{A} = |\boldsymbol{A}|\hat{\boldsymbol{A}}$.

In Cartesian coordinates, the three standard unit vectors are $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$, pointing along the positive x, y, z axes, respectively. These form a *right-handed orthonormal basis*: they are mutually perpendicular, each has unit length, and they satisfy $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$ (right-hand rule).

2.2 Vector Addition and Subtraction

Graphical Methods

Geometrically, vector addition follows the *head-to-tail rule*: to add $\boldsymbol{A} + \boldsymbol{B}$, place the tail of $\boldsymbol{B}$ at the head of $\boldsymbol{A}$; the sum runs from the tail of $\boldsymbol{A}$ to the head of $\boldsymbol{B}$. Equivalently, the *parallelogram rule*: draw $\boldsymbol{A}$ and $\boldsymbol{B}$ from the same point; the diagonal of the resulting parallelogram is $\boldsymbol{A} + \boldsymbol{B}$.

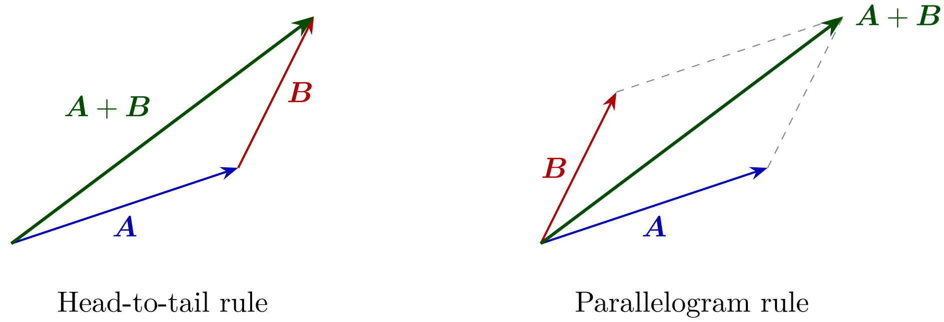


Figure 2.2.1: Two equivalent methods for graphical vector addition.

Vector addition is commutative ($\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$) and associative ($(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$).

The *negative* of a vector $\boldsymbol{A}$ is $-\boldsymbol{A}$: the vector with the same magnitude but opposite direction. **Vector subtraction** is defined as $\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+(-\boldsymbol{B})$. Geometrically, $\boldsymbol{A}-\boldsymbol{B}$ is the vector from the head of $\boldsymbol{B}$ to the head of $\boldsymbol{A}$ (when both are drawn from the same point).

Scalar multiplication: for a scalar c and vector $\boldsymbol{A}$, the product $c\boldsymbol{A}$ has magnitude $|c||\boldsymbol{A}|$ and points in the same direction as $\boldsymbol{A}$ if $c>0$, or the opposite direction if $c<0$.

2.3 Component Representation

In Cartesian coordinates, any vector in three-dimensional space can be written as

$$\boldsymbol{A}=A_x\hat{\boldsymbol{\imath}}+A_y\hat{\boldsymbol{\jmath}}+A_z\hat{\boldsymbol{k}}, \tag{2.2}$$

where A_x, A_y, A_z are the *components* (or *scalar projections*) of $\boldsymbol{A}$ onto each axis.

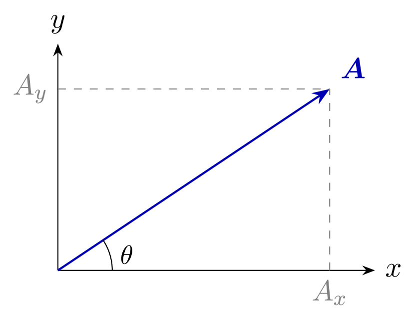


Figure 2.3.1: A vector $\boldsymbol{A}$ decomposed into its Cartesian components A_x and A_y. The angle θ is measured from the positive x-axis.

The magnitude is

$$|\boldsymbol{A}|=\sqrt{A_x^2+A_y^2+A_z^2}. \tag{2.3}$$

In two dimensions, if $\boldsymbol{A}$ makes angle θ with the positive x-axis:

$$A_x=|\boldsymbol{A}|\cos\theta, \qquad A_y=|\boldsymbol{A}|\sin\theta, \qquad \theta=\arctan\left(\frac{A_y}{A_x}\right). \tag{2.4}$$

Common Mistake 2.1: Arctangent Quadrant Ambiguity

When using $\theta = \arctan(A_y/A_x)$, the standard inverse tangent returns values in $(-\pi/2, \pi/2)$, covering only quadrants I and IV. If the vector lies in quadrant II or III (i.e., $A_x < 0$), you must add π to the result. Alternatively, use $\theta = \texttt{atan2}(A_y, A_x)$, which returns the correct angle in all four quadrants.

In component form, vector addition and scalar multiplication are performed component by component:

$$\boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\hat{\boldsymbol{\imath}} + (A_y + B_y)\hat{\boldsymbol{\jmath}} + (A_z + B_z)\hat{\boldsymbol{k}}, \qquad c\boldsymbol{A} = (cA_x)\hat{\boldsymbol{\imath}} + (cA_y)\hat{\boldsymbol{\jmath}} + (cA_z)\hat{\boldsymbol{k}}.$$

2.4 The Dot Product (Scalar Product)

Definition 2.2: Dot Product

The dot product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is the scalar

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \cos\theta, \tag{2.5}$$

where θ is the angle between the two vectors.

Equivalence of the geometric and component formulas. We can derive the component expression from the geometric one. Using $\boldsymbol{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}$ and $\boldsymbol{B} = B_x\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}} + B_z\hat{\boldsymbol{k}}$, expand by distributivity:

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}) + A_x B_y(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}) + \cdots$$

Since the basis vectors are orthonormal ($\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1$ and $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}} = 0$), all cross terms vanish and we obtain:

$$\boxed{\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z.} \tag{2.6}$$

This is the formula you will use most often in practice.

Properties: the dot product is commutative ($\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$), distributive ($\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$), and satisfies $\boldsymbol{A} \cdot \boldsymbol{A} = |\boldsymbol{A}|^2$. It is *not* associative (the expression $(\boldsymbol{A} \cdot \boldsymbol{B}) \cdot \boldsymbol{C}$ is meaningless, since $\boldsymbol{A} \cdot \boldsymbol{B}$ is a scalar).

Key Point 2.1: Perpendicularity Test

Two non-zero vectors are perpendicular if and only if $\boldsymbol{A} \cdot \boldsymbol{B} = 0$. The angle between them is

$$\theta = \arccos\left(\frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}\right).$$

Projection. The *scalar projection* of $\boldsymbol{B}$ onto the direction of $\boldsymbol{A}$ is $\text{comp}_{\boldsymbol{A}}\boldsymbol{B} = \boldsymbol{B} \cdot \hat{\boldsymbol{A}} = |\boldsymbol{B}| \cos\theta$. The *vector projection* is $\text{proj}_{\boldsymbol{A}}\boldsymbol{B} = (\boldsymbol{B} \cdot \hat{\boldsymbol{A}})\hat{\boldsymbol{A}}$. These are essential for decomposing forces along and perpendicular to a given direction.

Physical application: work. The work done by a constant force $\boldsymbol{F}$ over a displacement $\boldsymbol{d}$ is defined as $W = \boldsymbol{F} \cdot \boldsymbol{d} = |\boldsymbol{F}||\boldsymbol{d}| \cos\theta$. This is why you need the dot product in mechanics: work depends not just on the magnitudes of force and displacement, but on the angle between them. A force perpendicular to the displacement does zero work.

2.5 The Cross Product (Vector Product)

Definition 2.3: Cross Product

The cross product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is the vector

$$\boldsymbol{A} \times \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \sin\theta \, \hat{\boldsymbol{n}}, \tag{2.7}$$

where θ is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ (taken between 0 and π), and $\hat{\boldsymbol{n}}$ is the unit vector perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$, with direction determined by the **right-hand rule**.

The right-hand rule. Curl the fingers of your right hand from $\boldsymbol{A}$ toward $\boldsymbol{B}$ through the smaller angle between them. Your extended thumb points in the direction of $\boldsymbol{A} \times \boldsymbol{B}$. Equivalently: if a right-handed screw is turned from $\boldsymbol{A}$ toward $\boldsymbol{B}$, the screw advances in the direction of $\boldsymbol{A} \times \boldsymbol{B}$.

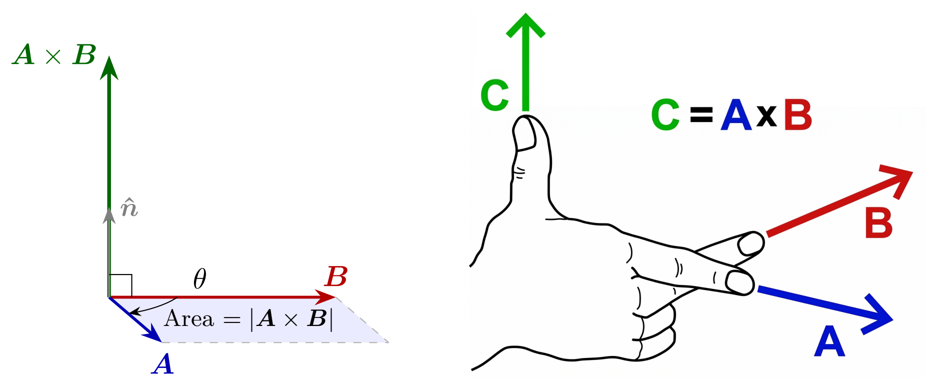


Figure 2.5.1: Left: the cross product $\boldsymbol{A} \times \boldsymbol{B}$ is perpendicular to the plane containing $\boldsymbol{A}$ and $\boldsymbol{B}$. Its magnitude equals the area of the shaded parallelogram. Right: the right-hand rule—point the index finger along $\boldsymbol{A}$ and the middle finger along $\boldsymbol{B}$; the thumb points in the direction of $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$.

The right-hand grip rule. A related convention arises whenever we represent a rotation by a vector. If an object rotates about an axis, we assign the angular velocity vector $\boldsymbol{\omega}$ along that axis using the *right-hand grip rule*: curl the fingers of the right hand in the direction of rotation, and the extended thumb points in the direction of $\boldsymbol{\omega}$. This same convention applies to angular momentum $\boldsymbol{L}$, torque $\boldsymbol{\tau}$, and any other quantity that describes rotational motion about an axis. The grip rule is not independent of the cross-product rule—it is a direct consequence of the definitions $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ and $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$—but it is often the more intuitive version to apply in rotational problems.

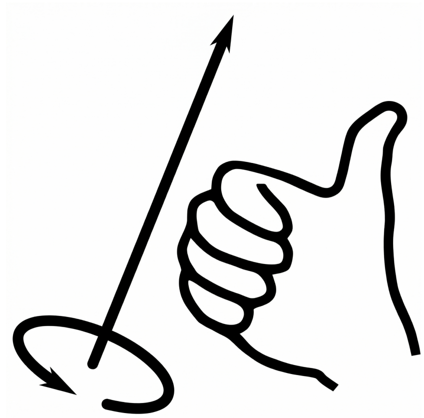


Figure 2.5.2: The right-hand grip rule: curl the fingers of the right hand in the direction of rotation; the thumb points along the angular velocity vector $\boldsymbol{\omega}$. The same rule determines the direction of angular momentum $\boldsymbol{L}$ and torque $\boldsymbol{\tau}$.

Component formula. Applying the cross product to each pair of basis vectors ($\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \mathbf{0}$) and expanding by distributivity gives:

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\hat{\boldsymbol{\imath}} - (A_x B_z - A_z B_x)\hat{\boldsymbol{\jmath}} + (A_x B_y - A_y B_x)\hat{\boldsymbol{k}}. \tag{2.8}$$

This is compactly written as a determinant:

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{2.9}$$

Key properties:

- *Anti-commutativity*: $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$. Reversing the order reverses the direction.
- *Self-cross*: $\boldsymbol{A} \times \boldsymbol{A} = \mathbf{0}$ (since $\sin 0 = 0$). Parallel vectors have zero cross product.
- *Geometric meaning*: $|\boldsymbol{A} \times \boldsymbol{B}|$ equals the area of the parallelogram spanned by $\boldsymbol{A}$ and $\boldsymbol{B}$.
- *Distributive*: $\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$.
- *Not associative*: in general, $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \neq (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$.

Proof that the cross product is perpendicular to both factors. We verify $(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{A} = 0$. Using the component formula (2.8):

$$\begin{aligned}(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{A} &= (A_y B_z - A_z B_y)A_x + (A_z B_x - A_x B_z)A_y + (A_x B_y - A_y B_x)A_z \\ &= A_x A_y B_z - A_x A_z B_y + A_y A_z B_x - A_x A_y B_z + A_x A_z B_y - A_y A_z B_x = 0.\end{aligned}$$

An identical calculation shows $(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{B} = 0$.

The Scalar Triple Product

The quantity $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$ is called the *scalar triple product*. It equals the (signed) volume of the parallelepiped defined by the three vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and can be computed as:

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{2.10}$$

The scalar triple product is unchanged under cyclic permutation: $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$. If this quantity is zero, the three vectors are coplanar.

The Lagrange Identity

An important identity connecting the dot and cross products is:

$$|\boldsymbol{A} \times \boldsymbol{B}|^2 + (\boldsymbol{A} \cdot \boldsymbol{B})^2 = |\boldsymbol{A}|^2 |\boldsymbol{B}|^2. \tag{2.11}$$

This follows immediately from $|\boldsymbol{A}|^2|\boldsymbol{B}|^2 \sin^2\theta + |\boldsymbol{A}|^2|\boldsymbol{B}|^2\cos^2\theta = |\boldsymbol{A}|^2|\boldsymbol{B}|^2$. It provides a useful check on cross-product calculations and is the vector analogue of the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

Physical applications of the cross product. The cross product appears throughout mechanics:

- *Torque*: $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$, the rotational effect of a force about a point.
- *Angular momentum*: $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, the rotational analogue of linear momentum.
- *Magnetic force*: $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$ (in electromagnetism).

In each case, the cross product captures the idea that only the *perpendicular* component of one vector relative to another produces the physical effect.

2.6 Coordinate Systems

Throughout this book, we use several coordinate systems, chosen to match the symmetry of the problem at hand.

2.6.1 Cartesian Coordinates

Cartesian coordinates (x, y, z) use fixed unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$. These are best when the motion has no special symmetry, or when the acceleration has constant components (e.g., projectile motion). The position vector is $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$.

2.6.2 Polar Coordinates (2D)

Polar coordinates (r, θ) describe a point in the plane by its distance r from the origin and its angle θ from the positive x-axis. The unit vectors are:

$$\hat{\boldsymbol{r}} = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}.$$

Unlike $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, these depend on θ and therefore change direction as the particle moves. The position vector is $\boldsymbol{r} = r\hat{\boldsymbol{r}}$. The *area element* is:

$$\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{2.12}$$

The factor of r arises because at distance r from the origin, an increment $\mathrm{d}\theta$ in angle corresponds to an arc length $r\,\mathrm{d}\theta$, not simply $\mathrm{d}\theta$. The full derivation of velocity and acceleration in polar coordinates is given in Chapter 4.

2.6.3 Cylindrical Coordinates

Cylindrical coordinates (ρ, ϕ, z) extend polar coordinates to three dimensions by adding the Cartesian z-axis. The relations to Cartesian coordinates are:

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z, \tag{2.13}$$

where $\rho \geq 0$ is the perpendicular distance from the z-axis and $\phi \in [0, 2\pi)$ is the azimuthal angle. The unit vectors are:

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\boldsymbol{\imath}} + \sin\phi\,\hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{\imath}} + \cos\phi\,\hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{z}} = \hat{\boldsymbol{k}}.$$

Note that $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ depend on ϕ (just like the 2D polar unit vectors), while $\hat{\boldsymbol{z}}$ is constant. The volume element is:

$$\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z. \tag{2.14}$$

Cylindrical coordinates are natural for problems with axial symmetry: rotating shafts, solenoids, and motion about a fixed axis.

2.6.4 Spherical Coordinates

Spherical coordinates (r, θ, ϕ) describe a point in 3D by its distance r from the origin, the polar angle θ measured from the positive z-axis $(0 \leq \theta \leq \pi)$, and the azimuthal angle ϕ measured from the positive x-axis in the xy-plane $(0 \leq \phi < 2\pi)$. The relations to Cartesian coordinates are:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta. \tag{2.15}$$

The three orthonormal unit vectors are:

$$\begin{aligned} \hat{\boldsymbol{r}} &= \sin\theta\cos\phi\,\hat{\boldsymbol{\imath}} + \sin\theta\sin\phi\,\hat{\boldsymbol{\jmath}} + \cos\theta\,\hat{\boldsymbol{k}}, \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\boldsymbol{\imath}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\jmath}} - \sin\theta\,\hat{\boldsymbol{k}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\boldsymbol{\imath}} + \cos\phi\,\hat{\boldsymbol{\jmath}}. \end{aligned} \tag{2.16}$$

All three unit vectors depend on position; only at a given point do they form a local orthonormal basis. The volume element is:

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{2.17}$$

The factor $r^2\sin\theta$ is the Jacobian of the transformation from Cartesian to spherical coordinates. The area element on a sphere of fixed radius r is $\mathrm{d}A = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$.

Spherical coordinates are the natural choice for problems with central symmetry: planetary orbits, gravitational and electric fields of point sources, and the hydrogen atom. The full derivation of the unit vectors, Jacobian, and volume element appears in Appendix B.

Common Mistake 2.2: Spherical Coordinate Conventions

Physicists and mathematicians use different conventions for spherical coordinates. In the physics convention used in this book, θ is the polar angle (from the z-axis) and ϕ is the azimuthal angle (in the xy-plane). Many mathematics textbooks reverse this notation. Always check the convention before using a formula from an unfamiliar source.

Worked Examples

Example 2.1 (Projection and decomposition). Decompose $\boldsymbol{F} = 5\hat{\boldsymbol{\imath}} + 12\hat{\boldsymbol{\jmath}}$ N into components parallel and perpendicular to the direction $\hat{\boldsymbol{n}} = \frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{\jmath}}$.

Solution. The parallel component is $F_{\parallel} = \boldsymbol{F} \cdot \hat{\boldsymbol{n}} = 5(3/5) + 12(4/5) = 3 + 9.6 = 12.6$ N. Thus $\boldsymbol{F}_{\parallel} = 12.6\,\hat{\boldsymbol{n}} = 7.56\hat{\boldsymbol{\imath}} + 10.08\hat{\boldsymbol{\jmath}}$ N. The perpendicular component is $\boldsymbol{F}_{\perp} = \boldsymbol{F} - \boldsymbol{F}_{\parallel} = (5 - 7.56)\hat{\boldsymbol{\imath}} + (12 - 10.08)\hat{\boldsymbol{\jmath}} = -2.56\hat{\boldsymbol{\imath}} + 1.92\hat{\boldsymbol{\jmath}}$ N. Check: $\boldsymbol{F}_{\parallel} \cdot \boldsymbol{F}_{\perp} = -19.35 + 19.35 = 0$. ✓

Example 2.2 (Angle between vectors). Find the angle between $\boldsymbol{A} = \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{B} = 3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{k}}$.

Solution. $\boldsymbol{A} \cdot \boldsymbol{B} = 3 + 0 + 8 = 11$. $|\boldsymbol{A}| = \sqrt{1+4+4} = 3$, $|\boldsymbol{B}| = \sqrt{9+16} = 5$. So $\cos\theta = 11/15$, giving $\theta = \arccos(11/15) \approx 42.8°$.

Example 2.3 (Cross product and area). Find the area of the triangle with vertices $P = (1,0,0)$, $Q = (0,2,0)$, $R = (0,0,3)$.

Solution. $\boldsymbol{PQ} = -\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$, $\boldsymbol{PR} = -\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{k}}$. $\boldsymbol{PQ} \times \boldsymbol{PR} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (6-0)\hat{\boldsymbol{\imath}} - (-3-0)\hat{\boldsymbol{\jmath}} + (0+2)\hat{\boldsymbol{k}} = 6\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$. Area of parallelogram $= |6\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}| = \sqrt{36+9+4} = 7$. Area of triangle $= 7/2 = 3.5$ square units.

Example 2.4 (Work as a dot product). A force $\boldsymbol{F} = (4\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}})$ N acts on a particle while it moves through displacement $\boldsymbol{d} = (2\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}})$ m. Find the work done.

Solution. $W = \boldsymbol{F} \cdot \boldsymbol{d} = 4(2) + (-3)(5) + 2(-1) = 8 - 15 - 2 = -9$ J. The negative sign means the force opposes the motion on average (the angle between $\boldsymbol{F}$ and $\boldsymbol{d}$ exceeds 90°).

Problems

Problem 2.1 ⋆

Given $\boldsymbol{A} = 4\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{B} = -2\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$, find: (a) $\boldsymbol{A} + \boldsymbol{B}$, (b) $|\boldsymbol{A}|$ and $|\boldsymbol{B}|$, (c) $\boldsymbol{A} \cdot \boldsymbol{B}$, (d) the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

Problem 2.2 ⋆⋆

Vector $\boldsymbol{C}$ has magnitude 10 and makes angle 120° with the positive x-axis. (a) Express $\boldsymbol{C}$ in component form. (b) Find a unit vector in the direction of $\boldsymbol{C}$. (c) Find a vector perpendicular to $\boldsymbol{C}$ with the same magnitude.

Problem 2.3 ⋆⋆⋆

Given $\boldsymbol{u} = 2\hat{\boldsymbol{\imath}} + \alpha\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$, $\boldsymbol{v} = \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$, $\boldsymbol{w} = 4\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \beta\hat{\boldsymbol{k}}$. (a) Find α such that $\boldsymbol{u} \perp \boldsymbol{v}$. (b) Find β such that $\boldsymbol{w} \perp \boldsymbol{v}$. (c) With these values, is $\boldsymbol{u} \perp \boldsymbol{w}$? Explain why perpendicularity is not transitive.

Problem 2.4 ⋆⋆⋆⋆

Prove: $|\boldsymbol{A} \times \boldsymbol{B}|^2 + (\boldsymbol{A} \cdot \boldsymbol{B})^2 = |\boldsymbol{A}|^2|\boldsymbol{B}|^2$. Verify for $\boldsymbol{A} = \hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{B} = 3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{k}}$.

Problem 2.5 ⋆⋆⋆⋆⋆

A particle's position $\boldsymbol{r}(t)$ satisfies $\boldsymbol{r} \cdot \boldsymbol{v} = 0$ for all time, where $\boldsymbol{v} = \dot{\boldsymbol{r}}$. (a) Prove that $|\boldsymbol{r}|$ is constant. (b) What path does the particle follow? (c) If $\boldsymbol{r} \times \boldsymbol{v} = \boldsymbol{L}$ (a constant vector), show that the motion is confined to a plane.

Problem 2.6 ⋆⋆

Find a unit vector that bisects the angle between $\boldsymbol{A} = 3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ and $\boldsymbol{B} = 4\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$.

Problem 2.7 ⋆⋆⋆

The position of a particle is $\boldsymbol{r}(t) = (3t^2 - 1)\hat{\boldsymbol{\imath}} + (4t)\hat{\boldsymbol{\jmath}} + (2t^3)\hat{\boldsymbol{k}}$ (in meters, t in seconds). (a) Find the velocity and acceleration at $t = 1\,\text{s}$. (b) Find the magnitude of the velocity at $t = 1\,\text{s}$. (c) Find the angle between $\boldsymbol{v}$ and $\boldsymbol{a}$ at $t = 1\,\text{s}$.

Problem 2.8 ⋆⋆⋆

Show that the scalar triple product $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$ gives the volume of the parallelepiped defined by $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$. Compute for $\boldsymbol{A} = \hat{\boldsymbol{\imath}}$, $\boldsymbol{B} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$, $\boldsymbol{C} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$.

Problem 2.9 ⋆⋆⋆⋆

The vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ have magnitudes A and B. Show that $|\boldsymbol{A} + \boldsymbol{B}|^2 + |\boldsymbol{A} - \boldsymbol{B}|^2 = 2(A^2 + B^2)$ (the parallelogram law). Give a geometric interpretation.

Problem 2.10 ⋆⋆

Decompose $\boldsymbol{F} = 5\hat{\boldsymbol{\imath}} + 12\hat{\boldsymbol{\jmath}}$ N into components parallel and perpendicular to the direction $\hat{\boldsymbol{n}} = (3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}})/5$.

Chapter 3

Kinematics in One Dimension

Kinematics is the study of motion without reference to its causes: we describe *how* objects move, deferring the question of *why* to Newton's laws in Chapter 5. We begin with the simplest case: motion along a straight line. Despite its apparent simplicity, one-dimensional kinematics contains all the essential ideas (position, velocity, acceleration, integration of equations of motion) that generalize directly to two and three dimensions.

3.1 Position, Displacement, and Velocity

Definition 3.1: Position, Displacement, Velocity

Position $x(t)$: the location of a particle on a coordinate axis at time t.
Displacement: $\Delta x = x_f - x_i$ (a signed quantity—positive means motion in the $+x$ direction).
Distance: the total path length traveled (always non-negative).
Average velocity: $\bar{v} = \Delta x/\Delta t$ (displacement per unit time).
Average speed: $\bar{s} =$ (total distance)$/\Delta t$ (always ≥ 0).
Instantaneous velocity: $v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{\mathrm{d}x}{\mathrm{d}t}$.
Speed: $|v(t)|$, the magnitude of the velocity.

The distinction between displacement and distance is important: a particle that moves 3 m to the right and then 1 m to the left has displacement $\Delta x = +2$ m but distance traveled $d = 4$ m. Similarly, the average velocity over this trip is $\bar{v} = 2/\Delta t$, but the average speed is $\bar{s} = 4/\Delta t$—they are not the same quantity.

Instantaneous velocity as a limit. The definition $v = \mathrm{d}x/\mathrm{d}t$ means: take the ratio $\Delta x/\Delta t$ over progressively shorter time intervals centered around time t and see what value it approaches. Geometrically, this is the slope of the tangent line to the $x(t)$ curve at time t.

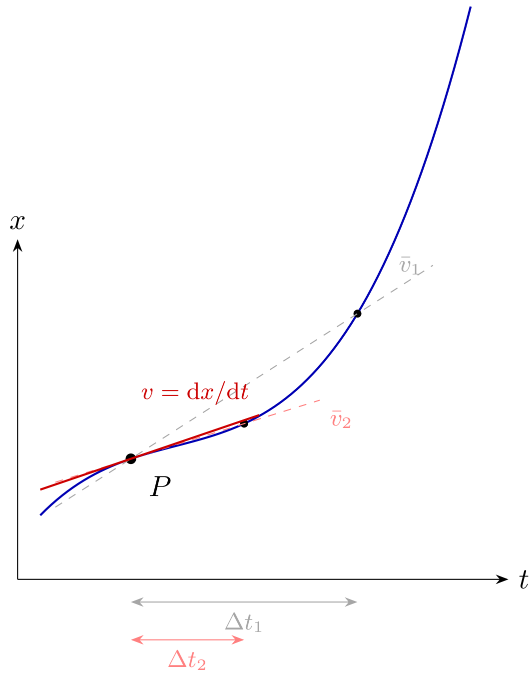


Figure 3.1.1: As the time interval Δt shrinks, the secant line (average velocity $\bar{v} = \Delta x/\Delta t$) approaches the tangent line (instantaneous velocity $v = \mathrm{d}x/\mathrm{d}t$) at point P.

When we say "the velocity at time t," we always mean this instantaneous value unless explicitly stated otherwise. The instantaneous velocity can be positive (motion in the $+x$ direction), negative (motion in the $-x$ direction), or zero (the particle is momentarily at rest, though it may still be accelerating).

Graphical Interpretation

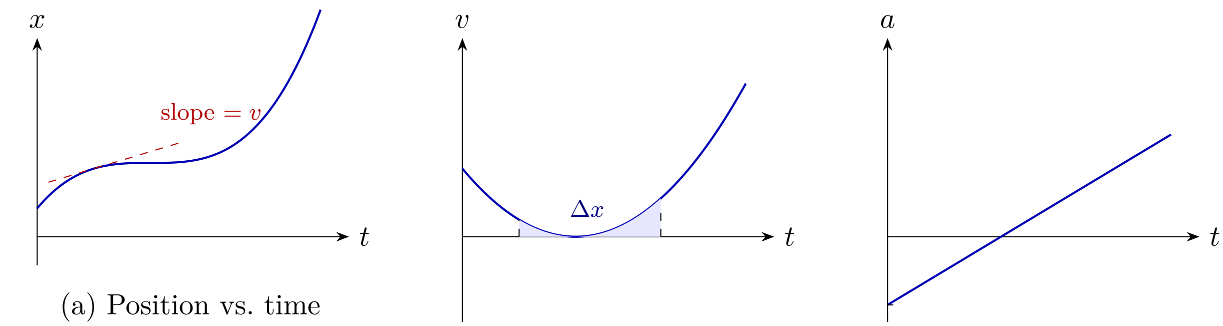
The connections between position, velocity, and acceleration graphs are central to kinematics. On a position-vs-time graph, the instantaneous velocity equals the *slope*:

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \text{slope of } x(t).$$

Conversely, on a velocity-vs-time graph, the displacement over $[t_1, t_2]$ equals the *signed area* under the curve:

$$\Delta x = \int_{t_1}^{t_2} v(t)\,\mathrm{d}t. \tag{3.1}$$

Similarly, $a(t) = \mathrm{d}v/\mathrm{d}t$ is the slope of the $v(t)$ graph, and $\Delta v = \int a\,\mathrm{d}t$ is the area under $a(t)$.



(a) Position vs. time

(b) Velocity vs. time

(c) Acceleration vs. time

Figure 3.1.2: The three kinematic graphs and their relationships. The slope of (a) gives (b); the slope of (b) gives (c). The area under (b) gives displacement; the area under (c) gives the change in velocity.

These relationships are exact: they are simply the geometric interpretation of the fundamental theorem of calculus applied to the definitions $v = \mathrm{d}x/\mathrm{d}t$ and $a = \mathrm{d}v/\mathrm{d}t$.

3.2 Acceleration

Definition 3.2: Acceleration

Average acceleration: $\bar{a} = \Delta v/\Delta t$.
Instantaneous acceleration: $a = \mathrm{d}v/\mathrm{d}t = \mathrm{d}^2x/\mathrm{d}t^2$.

Acceleration describes how quickly the velocity is changing. A common source of confusion is the relationship between the signs of velocity and acceleration:

- If v and a have the *same sign*, the object is speeding up (the speed $|v|$ is increasing).
- If v and a have *opposite signs*, the object is slowing down (the speed $|v|$ is decreasing).

"Deceleration" is not a separate physical concept: it simply means the acceleration vector opposes the velocity vector. A car moving in the $-x$ direction with a positive acceleration is *slowing down*, not speeding up.

3.3 Constant Acceleration Equations

When the acceleration a is constant, the equations of motion can be derived by direct integration.

Step 1. From $a = \mathrm{d}v/\mathrm{d}t$, separate and integrate: $\int_{v_0}^{v} \mathrm{d}v' = \int_0^t a\,\mathrm{d}t'$, giving:

$$\boxed{v = v_0 + at.} \tag{3.2}$$

Step 2. From $v = \mathrm{d}x/\mathrm{d}t$, substitute (3.2): $\mathrm{d}x = (v_0 + at)\,\mathrm{d}t$. Integrate from 0 to t:

$$\boxed{x = x_0 + v_0 t + \tfrac{1}{2}at^2.} \tag{3.3}$$

Step 3. To eliminate t, use the chain rule: $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$, so $a\,\mathrm{d}x = v\,\mathrm{d}v$. Integrate: $\int_{x_0}^{x} a\,\mathrm{d}x' = \int_{v_0}^{v} v'\,\mathrm{d}v'$, giving:

$$\boxed{v^2 = v_0^2 + 2a(x - x_0).} \tag{3.4}$$

Step 4. Alternatively, note that for constant acceleration the average velocity is $\bar{v} = \frac{1}{2}(v_0 + v)$, so:

$$\boxed{x = x_0 + \tfrac{1}{2}(v_0 + v)\,t.} \tag{3.5}$$

Key Point 3.1: Choosing the Right Equation

Each kinematic equation involves four of the five variables x, v, v_0, a, t. To solve a problem: identify which three quantities are known and which one is desired, then choose the equation that contains exactly those four. Eq. (3.2) omits x; Eq. (3.3) omits v; Eq. (3.4) omits t; Eq. (3.5) omits a.

Common Mistake 3.1: Constant Acceleration Only

Equations (3.2)–(3.5) are valid *only* when the acceleration is constant. If a varies with time, you must return to the definitions $v = \mathrm{d}x/\mathrm{d}t$ and $a = \mathrm{d}v/\mathrm{d}t$ and integrate directly, as in Example 3.3.

General Methods for Non-Constant Acceleration

When a is not constant, the kinematic equations above do not apply. There are three standard cases, each requiring a different integration technique:

Case 1: $a = a(t)$ **(acceleration given as a function of time).** Integrate directly:

$$v(t) = v_0 + \int_0^t a(t')\,\mathrm{d}t', \qquad x(t) = x_0 + \int_0^t v(t')\,\mathrm{d}t'.$$

Case 2: $a = a(v)$ **(acceleration depends on velocity).** Separate variables using $a = \mathrm{d}v/\mathrm{d}t$:

$$\mathrm{d}t = \frac{\mathrm{d}v}{a(v)} \implies t = \int_{v_0}^{v} \frac{\mathrm{d}v'}{a(v')}.$$

This arises naturally in drag problems, where the resistive force (and hence the acceleration) depends on speed. Alternatively, using $a = v\,\mathrm{d}v/\mathrm{d}x$:

$$\mathrm{d}x = \frac{v\,\mathrm{d}v}{a(v)} \implies x - x_0 = \int_{v_0}^{v} \frac{v'\,\mathrm{d}v'}{a(v')}.$$

Case 3: $a = a(x)$ **(acceleration depends on position).** Use the chain rule identity $a = v\,\mathrm{d}v/\mathrm{d}x$:

$$a(x)\,\mathrm{d}x = v\,\mathrm{d}v \implies \int_{x_0}^{x} a(x')\,\mathrm{d}x' = \tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2.$$

This gives v as a function of x directly. We will see in Chapter 7 that this is equivalent to the work-energy theorem.

The identity $a = v\,\mathrm{d}v/\mathrm{d}x$ (used already in Step 3 of the constant-acceleration derivation) is one of the most powerful tools in one-dimensional kinematics. It eliminates time entirely, relating velocity directly to position.

3.4 Free Fall

Near Earth's surface, all objects (neglecting air resistance) experience a constant downward acceleration of magnitude $g \approx 9.8\,\text{m/s}^2$. This remarkable fact—that the acceleration due to gravity is independent of an object's mass, composition, or shape—was first demonstrated experimentally by Galileo Galilei around 1600, overturning the Aristotelian doctrine that heavier objects fall faster. Galileo used inclined planes to "dilute" gravity (slowing the motion enough to measure with the instruments of his time), and showed that the distance fallen is proportional to t^2—the hallmark of constant acceleration.

With the y-axis pointing upward, $a_y = -g$ (the minus sign because gravity acts downward while the axis points up). The kinematic equations apply with the replacements $x \to y$, $a \to -g$:

$$\begin{aligned} v_y &= v_{0y} - gt, \\ y &= y_0 + v_{0y}\,t - \tfrac{1}{2}gt^2, \\ v_y^2 &= v_{0y}^2 - 2g(y - y_0). \end{aligned}$$

Symmetry of vertical motion. For a projectile launched straight up from ground level: the time to reach maximum height equals the time to fall back down; the speed at any height on the way up equals the speed at the same height on the way down; and the acceleration is $-g$ throughout the entire flight—including at the peak, where $v = 0$ but $a = -g$ (a common conceptual pitfall for students who confuse zero velocity with zero acceleration).

Worked Examples

Example 3.1 (Braking distance). A car traveling at $v_0 = 30\,\text{m/s}$ brakes uniformly and stops in $60\,\text{m}$. Find the deceleration and stopping time.

Solution. From $v^2 = v_0^2 + 2a\Delta x$: $0 = 900 + 2a(60)$, so $a = -7.5\,\text{m/s}^2$. From $v = v_0 + at$: $0 = 30 - 7.5t$, so $t = 4.0\,\text{s}$.

Note: if the initial speed doubles to $60\,\text{m/s}$, the stopping distance quadruples to $240\,\text{m}$ (since $\Delta x \propto v_0^2$ at constant deceleration). This is why speed limits have such large safety implications.

Example 3.2 (Thrown ball with free fall). A ball is thrown vertically upward at $20\,\text{m/s}$. Find (a) the maximum height, (b) the time to reach it, (c) the velocity when it returns to the starting point.

Solution. (a) At max height, $v = 0$: $0 = 20^2 - 2(9.8)h \Rightarrow h = 20.4\,\text{m}$.

(b) $0 = 20 - 9.8t \Rightarrow t = 2.04\,\text{s}$.

(c) By symmetry (or by $v^2 = v_0^2 + 2a\Delta x$ with $\Delta x = 0$): $v = -20\,\text{m/s}$ (downward at the same speed it was launched).

Example 3.3 (Non-constant acceleration). A particle moves along the x-axis with acceleration $a(t) = 6t\ \text{m/s}^2$ and initial conditions $v(0) = 2\,\text{m/s}$, $x(0) = 1\,\text{m}$. Find $v(t)$ and $x(t)$.

Solution. Since a is not constant, we integrate directly:

$$v(t) = v_0 + \int_0^t 6t'\,\mathrm{d}t' = 2 + 3t^2.$$

$$x(t) = x_0 + \int_0^t v(t')\,\mathrm{d}t' = 1 + \int_0^t (2 + 3t'^2)\,\mathrm{d}t' = 1 + 2t + t^3.$$

At $t = 2\,\text{s}$: $v = 2 + 12 = 14\,\text{m/s}$, $x = 1 + 4 + 8 = 13\,\text{m}$.

Example 3.4 (Two-car meeting problem). Car A starts from rest at the origin with constant acceleration $a_A = 3.0\,\mathrm{m/s^2}$. Two seconds later, car B passes the origin moving at constant velocity $v_B = 20\,\mathrm{m/s}$. When and where does A overtake B?

Solution. Let t be measured from when A starts. Then $x_A = \frac{1}{2}(3.0)t^2 = 1.5t^2$ and $x_B = 20(t-2)$ for $t \geq 2$. Setting $x_A = x_B$: $1.5t^2 = 20t - 40$, or $1.5t^2 - 20t + 40 = 0$, giving $t = \frac{20 \pm \sqrt{400-240}}{3} = \frac{20 \pm 12.65}{3}$. So $t = 10.9$ s or $t = 2.45$ s. The first meeting at $t = 2.45$ s is when B catches up to A; the second at $t = 10.9$ s is when A overtakes B. At $t = 10.9$ s: $x = 1.5(10.9)^2 \approx 178\,\mathrm{m}$.

Example 3.5 (Graphical analysis). A particle's velocity is given by: $v = 10\,\mathrm{m/s}$ for $0 \leq t < 2\,\mathrm{s}$, then v decreases linearly to $-5\,\mathrm{m/s}$ at $t = 5\,\mathrm{s}$, then remains at $-5\,\mathrm{m/s}$ until $t = 7\,\mathrm{s}$. Find the displacement and distance traveled over $[0, 7\,\mathrm{s}]$.

Solution. The displacement is the net signed area under the v-t graph. For $0 \leq t < 2$: area $= 10 \times 2 = 20$ m. For $2 \leq t < 5$: this is a trapezoid with bases 10 and -5 and width 3: area $= \frac{1}{2}(10 + (-5))(3) = 7.5$ m. For $5 \leq t \leq 7$: area $= (-5)(2) = -10$ m. Total displacement: $20 + 7.5 - 10 = 17.5\,\mathrm{m}$.

For the distance, we need $\int |v|\,\mathrm{d}t$. The velocity crosses zero at some time in $[2, 5]$: from $v = 10 - 5(t-2) = 0$ we get $t = 4$ s. Positive area (0 to 4 s): $20 + \frac{1}{2}(10)(2) = 20 + 10 = 30$ m. Negative area (4 to 7 s): $\frac{1}{2}(5)(1) + 5(2) = 2.5 + 10 = 12.5$ m. Total distance: $30 + 12.5 = 42.5\,\mathrm{m}$.

Position and Velocity from Graphs

Example 3.6 (Reaction time and braking). A driver traveling at $v_0 = 25\,\mathrm{m/s}$ ($\approx 56\,\mathrm{mph}$) sees a hazard and brakes. The reaction time (delay before braking begins) is $t_r = 0.7\,\mathrm{s}$, and the braking deceleration is $a = -8.0\,\mathrm{m/s^2}$. Find the total stopping distance.

Solution. During the reaction time, the car travels at constant velocity: $d_r = v_0 t_r = 25(0.7) = 17.5\,\mathrm{m}$. During braking: $v^2 = v_0^2 + 2a\,d_b$ with $v = 0$, so $d_b = v_0^2/(2|a|) = 625/16 = 39.1\,\mathrm{m}$. Total stopping distance: $d = 17.5 + 39.1 = 56.6\,\mathrm{m}$.

Note: the reaction distance (17.5 m) is a substantial fraction of the total. At higher speeds, the braking distance grows as v_0^2 while the reaction distance grows only as v_0, so braking dominates. But the reaction-time contribution is always present and irreducible; this is why distracted driving is so dangerous.

Example 3.7 (Object dropped from a rising balloon). A balloon ascends at a constant $5.0\,\mathrm{m/s}$. At height $h = 40\,\mathrm{m}$, a sandbag is released. Find (a) the maximum height of the sandbag, (b) the time to reach the ground, and (c) the speed at impact.

Solution. At the moment of release, the sandbag has the balloon's upward velocity: $v_0 = +5.0\,\mathrm{m/s}$ and $y_0 = 40\,\mathrm{m}$. After release, $a = -g = -9.8\,\mathrm{m/s^2}$.

(a) At max height: $v^2 = v_0^2 - 2g(y_{\max} - y_0) = 0$, so $y_{\max} = y_0 + v_0^2/(2g) = 40 + 25/19.6 = 41.3\,\mathrm{m}$.

(b) When the sandbag hits the ground, $y = 0$: $0 = 40 + 5t - 4.9t^2$. Using the quadratic formula: $t = \frac{5 + \sqrt{25 + 784}}{9.8} = \frac{5 + 28.44}{9.8} = 3.41\,\mathrm{s}$ (taking the positive root).

(c) $v = v_0 - gt = 5 - 9.8(3.41) = -28.4\,\mathrm{m/s}$ (downward). The impact speed is $28.4\,\mathrm{m/s}$.

Since $v = \mathrm{d}x/\mathrm{d}t$, the velocity equals the slope of the $x(t)$ graph. Conversely, the displacement equals the area under the $v(t)$ curve. Similarly, $a = \mathrm{d}v/\mathrm{d}t$ means the acceleration is the slope of $v(t)$, and $\Delta v = \int a\,\mathrm{d}t$ is the area under $a(t)$.

This graphical interpretation is extremely useful when the motion is piecewise-defined (e.g., different constant accelerations in different time intervals). It also provides a powerful consistency check: if you compute $x(t)$ from given $v(t)$ data, the slope of your resulting $x(t)$ curve should match the original $v(t)$ at every point.

Problems

Problem 3.1 ⋆

A car accelerates from rest to 25 m/s in 8.0 s. Find (a) the acceleration, (b) the distance traveled.

Problem 3.2 ⋆⋆

A ball is dropped from height H. Simultaneously, another ball is thrown upward from the ground with speed v_0. (a) At what height do they meet? (b) What is each ball's velocity when they meet? (c) For what v_0 do they meet at $H/2$?

Problem 3.3 ⋆⋆⋆

Car A starts from rest at $x = 0$ with $a_A = 3.0\,\mathrm{m/s^2}$. At $t = 2.0\,\mathrm{s}$, car B passes the origin at constant $v_B = 20\,\mathrm{m/s}$. (a) Write $x_A(t)$ and $x_B(t)$. (b) When are they at the same position? (c) When do they have the same velocity? (d) What is the maximum separation?

Problem 3.4 ⋆⋆⋆⋆

A particle has velocity $v(t) = v_0 - \beta t^2$ with $x(0) = 0$. (a) Find $a(t)$ and $x(t)$. (b) When does it return to the origin? (c) What is the total distance traveled before returning?

Problem 3.5 ⋆⋆⋆⋆⋆

A rocket has acceleration $a(t) = a_0 - kt$ for $0 \le t \le t_1 = a_0/k$, then the engine cuts off and the rocket is subject to gravity alone. (a) Find $v(t)$ and $x(t)$ during powered flight. (b) Find the maximum height in terms of a_0, k, g. (c) Calculate for $a_0 = 30\,\mathrm{m/s^2}$, $k = 3\,\mathrm{m/s^3}$, $g = 10\,\mathrm{m/s^2}$.

Problem 3.6 ⋆⋆

A stone is thrown vertically upward at 20 m/s. (a) Maximum height. (b) Time in the air. (c) Speed when it returns. ($g = 10\,\mathrm{m/s^2}$.)

Problem 3.7 ⋆⋆⋆

A particle has acceleration $a(t) = 6t\ \mathrm{m/s^2}$. At $t = 0$: $v = 2\,\mathrm{m/s}$, $x = 1\,\mathrm{m}$. Find $v(t)$, $x(t)$, and the position at $t = 2\,\mathrm{s}$.

Problem 3.8 ⋆⋆

Two trains approach each other on the same track, each at 30 m/s, when they are 500 m apart. Both brake with $a = -3\,\mathrm{m/s^2}$. Do they collide?

Problem 3.9 ⋆⋆⋆⋆

A police car at rest starts accelerating at $a_p = 3\,\mathrm{m/s^2}$ at the instant a speeder passes at constant $v_s = 30\,\mathrm{m/s}$. (a) When does the police car catch up? (b) How fast is the police car going? (c) How far has it traveled?

Problem 3.10 ⋆⋆⋆

An elevator accelerates upward at $2.0\,\mathrm{m/s^2}$ from rest. A bolt falls from the ceiling (height 3.0 m above the floor) 2.0 s after the elevator starts. (a) How long until the bolt hits the floor? (Hint: work in the elevator frame.) (b) How far has the bolt fallen in the ground frame?

Chapter 4

Kinematics in Two and Three Dimensions

In Chapter 3 we described motion along a straight line. Most real motion, however, takes place in two or three dimensions: a thrown ball follows a parabolic arc, a satellite traces an ellipse, a car rounds a curve. In this chapter we extend the kinematic framework to multiple dimensions using vector notation, and we develop the key special cases of projectile motion and circular motion.

4.1 Position, Velocity, and Acceleration Vectors

Definition 4.1: Kinematic Vectors

Position: $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$.
Velocity: $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}}$.
Acceleration: $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}$.

A crucial insight: *the component motions are independent*. The x-component of the motion is governed solely by F_x, and the y-component solely by F_y.

4.2 Projectile Motion

When the only force is gravity ($\boldsymbol{a} = -g\hat{\boldsymbol{\jmath}}$ with upward positive), the horizontal and vertical motions decouple completely.

Theorem 4.1: Projectile Equations

For initial velocity v_0 at angle θ above horizontal:

$$x(t) = x_0 + (v_0 \cos\theta)\, t, \tag{4.1}$$

$$y(t) = y_0 + (v_0 \sin\theta)\, t - \tfrac{1}{2} g t^2, \tag{4.2}$$

$$v_x = v_0 \cos\theta \quad \text{(constant)}, \tag{4.3}$$

$$v_y(t) = v_0 \sin\theta - gt. \tag{4.4}$$

For launch from ground level ($y_0 = 0$):

- Time of flight: $T = 2v_0 \sin\theta/g$.
- Maximum height: $H = v_0^2 \sin^2\theta/(2g)$.
- Range: $R = v_0^2 \sin(2\theta)/g$.

Key Point 4.1: Maximum Range and Complementary Angles

Maximum range occurs at $\theta = 45°$. Complementary angles (θ and $90° - \theta$) give equal range: the shallow trajectory is fast and low, while the steep trajectory is slow and high.

Trajectory equation. Eliminating t between (4.1) and (4.2):

$$y = x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta}. \tag{4.5}$$

This is a downward-opening parabola, confirming the familiar shape of a projectile trajectory.

Derivation of the range formula. Setting $y = 0$ in (4.2) with $y_0 = 0$: $0 = v_0 \sin\theta\, t - \frac{1}{2}gt^2 = t(v_0\sin\theta - \frac{1}{2}gt)$. The nonzero solution is $T = 2v_0\sin\theta/g$. Substituting into (4.1):

$$R = v_0\cos\theta \cdot \frac{2v_0\sin\theta}{g} = \frac{v_0^2 \cdot 2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin(2\theta)}{g},$$

where we used the double-angle identity $2\sin\theta\cos\theta = \sin(2\theta)$. Since $\sin(2\theta)$ is maximized when $2\theta = 90°$, the maximum range occurs at $\theta = 45°$.

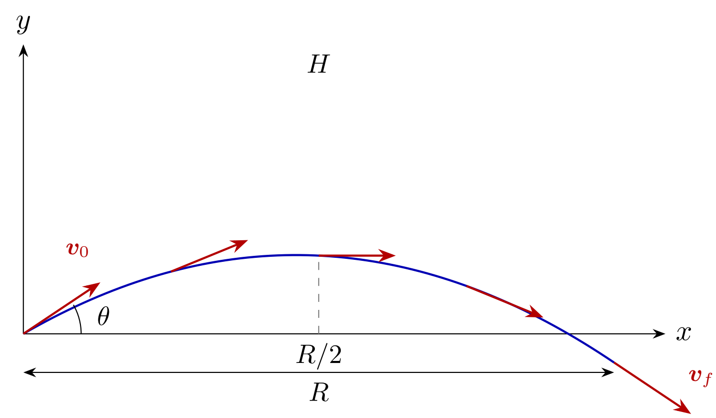


Figure 4.2.1: Projectile trajectory showing velocity vectors (red) at five equally spaced times. The horizontal component v_x remains constant; the vertical component v_y decreases linearly. At the peak, $v_y = 0$ and the velocity is purely horizontal.

Worked Examples

Example 4.1 (Monkey and hunter). A hunter aims directly at a monkey hanging from a tree at height h and horizontal distance d. At the instant the gun fires, the monkey lets go and falls. Show that the bullet hits the monkey regardless of the launch speed (provided it reaches the monkey before hitting the ground).

Solution. The bullet, aimed at angle $\theta = \arctan(h/d)$, has trajectory:

$$y_b = x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta}.$$

At $x = d$: $y_b = h - \frac{gd^2}{2v_0^2\cos^2\theta}$.

The monkey's height at time t is $y_m = h - \frac{1}{2}gt^2$. The bullet reaches $x = d$ at $t = d/(v_0\cos\theta)$:

$$y_m = h - \frac{g}{2}\left(\frac{d}{v_0\cos\theta}\right)^2 = h - \frac{gd^2}{2v_0^2\cos^2\theta} = y_b.$$

The bullet and monkey are at the same height at the same time: both "fall" the same distance $\frac{1}{2}gt^2$ below the straight-line aim. This is a vivid demonstration that horizontal and vertical motions are independent: gravity affects the bullet's trajectory in exactly the same way it affects the monkey's free fall.

Example 4.2 (Range on an inclined plane). A projectile is launched at speed v_0 and angle α from the base of a hill inclined at angle β to the horizontal. Find the range along the hill.

Solution. Using coordinates with x along the hill and y perpendicular to it (or simply rotating the standard equations), the range along the hill is:

$$R = \frac{2v_0^2 \sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta}.$$

This is maximized when $\alpha = \pi/4 + \beta/2$ (the bisector of the angle between the hill surface and the vertical). Setting $\beta = 0$ recovers $R = v_0^2 \sin(2\alpha)/g$.

Example 4.3 (Projectile off a cliff). A ball is thrown horizontally at $v_0 = 20\,\text{m/s}$ from a cliff of height $h = 45\,\text{m}$.

(a) Time of flight: $h = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{2h/g} = \sqrt{90/9.8} = 3.03\,\text{s}$.

(b) Range: $R = v_0 t = 20 \times 3.03 = 60.6\,\text{m}$.

(c) Impact angle: $v_y = gt = 29.7$ m/s, $v_x = 20$ m/s, so $\phi = \arctan(29.7/20) = 56.0°$ below horizontal.

(d) Impact speed: $v = \sqrt{v_x^2 + v_y^2} = \sqrt{400 + 882} = 35.8\,\text{m/s}$. Equivalently by energy: $v = \sqrt{v_0^2 + 2gh} = \sqrt{400 + 882} = 35.8\,\text{m/s}$.

4.3 Polar Coordinates and Circular Motion

Many problems in mechanics (especially those involving rotation) are most naturally described in polar coordinates (r, θ).

4.3.1 Polar Unit Vectors

At every point, we define:

$$\hat{\boldsymbol{r}} = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}.$$

The vector $\hat{\boldsymbol{r}}$ points radially outward; $\hat{\boldsymbol{\theta}}$ points in the direction of increasing θ. Both have unit magnitude and are mutually perpendicular: $\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\theta}} = 0$.

Unlike $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, the polar unit vectors depend on θ and therefore change with time. Their time derivatives are:

$$\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = \dot{\theta}\,\hat{\boldsymbol{\theta}}, \qquad \frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\dot{\theta}\,\hat{\boldsymbol{r}}. \tag{4.6}$$

To see this, differentiate the Cartesian definitions directly:

$$\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) = -\dot{\theta}\sin\theta\,\hat{\boldsymbol{\imath}} + \dot{\theta}\cos\theta\,\hat{\boldsymbol{\jmath}} = \dot{\theta}(-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}) = \dot{\theta}\,\hat{\boldsymbol{\theta}}.$$

Similarly:

$$\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}) = -\dot{\theta}\cos\theta\,\hat{\boldsymbol{\imath}} - \dot{\theta}\sin\theta\,\hat{\boldsymbol{\jmath}} = -\dot{\theta}(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) = -\dot{\theta}\,\hat{\boldsymbol{r}}.$$

Note the structure: differentiating $\hat{\boldsymbol{r}}$ rotates it by 90° to give $\hat{\boldsymbol{\theta}}$, and differentiating $\hat{\boldsymbol{\theta}}$ rotates it by another 90° to give $-\hat{\boldsymbol{r}}$. This is a consequence of the rotation matrix $R(\theta)$; the polar unit vectors are related to the Cartesian ones by a θ-dependent rotation, so their time derivatives pick up the angular velocity $\dot{\theta}$.

4.3.2 Position, Velocity, and Acceleration in Polar Coordinates

Starting from $\boldsymbol{r} = r\hat{\boldsymbol{r}}$ and applying the product rule:

$$\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{\boldsymbol{r}}) = \dot{r}\hat{\boldsymbol{r}} + r\dot{\hat{\boldsymbol{r}}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

Differentiating once more:

$$\begin{aligned}\boldsymbol{a} &= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\hat{\boldsymbol{r}}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}} \\ &= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}(-\dot{\theta}\hat{\boldsymbol{r}}) \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned}$$

The first component is the radial acceleration; the second is the transverse (angular) acceleration. The term $-r\dot{\theta}^2$ is the centripetal acceleration (directed inward), and the term $2\dot{r}\dot{\theta}$ is the Coriolis-like term that arises when the radius changes during rotation.

Theorem 4.2: Polar Kinematics

$$\boldsymbol{r} = r\,\hat{\boldsymbol{r}}, \tag{4.7}$$
$$\boldsymbol{v} = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}, \tag{4.8}$$
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}}. \tag{4.9}$$

The derivation proceeds by differentiating $\boldsymbol{r} = r\hat{\boldsymbol{r}}$ twice, using the product rule and (4.6) at each step. A full derivation is given in Appendix B.5.

In the acceleration (4.9), the radial term $-r\dot{\theta}^2$ is the centripetal acceleration (directed inward), and the tangential term $2\dot{r}\dot{\theta}$ arises from the curvilinear coordinate system. In the inertial frame, this $2\dot{r}\dot{\theta}$ term is purely kinematic: it is *not* a Coriolis force, which only appears in non-inertial rotating reference frames.

4.3.3 Uniform Circular Motion

For motion in a circle of constant radius r with constant angular velocity $\omega = \dot{\theta}$, we have $\dot{r} = 0$, $\ddot{r} = 0$, $\ddot{\theta} = 0$. The kinematics simplify to:

$$\boldsymbol{v} = r\omega\,\hat{\boldsymbol{\theta}}, \qquad \boldsymbol{a} = -r\omega^2\,\hat{\boldsymbol{r}} = -\frac{v^2}{r}\hat{\boldsymbol{r}}. \tag{4.10}$$

The acceleration is purely centripetal, directed toward the center of the circle with magnitude $a_c = v^2/r = r\omega^2$.

Period and frequency. The time for one complete revolution is the *period*:

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}. \tag{4.11}$$

The *frequency* $f = 1/T$ counts revolutions per unit time. The angular velocity is $\omega = 2\pi f$.

Geometric derivation of centripetal acceleration. Even without polar coordinates, one can derive $a_c = v^2/r$ from a purely geometric argument. Consider a particle moving at constant speed v around a circle of radius r. In a small time Δt, the particle sweeps out angle $\Delta\theta = v\Delta t/r$. The velocity vector, always tangent to the circle, also rotates by $\Delta\theta$. The change in velocity $|\Delta\boldsymbol{v}| = v\Delta\theta = v^2\Delta t/r$ (using the small-angle approximation $|\Delta\boldsymbol{v}| \approx v\Delta\theta$ for the isosceles triangle formed by $\boldsymbol{v}(t)$, $\boldsymbol{v}(t+\Delta t)$, and $\Delta\boldsymbol{v}$). Taking the limit:

$$a = \lim_{\Delta t\to 0} \frac{|\Delta\boldsymbol{v}|}{\Delta t} = \frac{v^2}{r}.$$

The direction of $\Delta\boldsymbol{v}$ points toward the center of the circle (it is the "inward" bisector of the two velocity vectors), confirming that the acceleration is centripetal.

4.3.4 Non-Uniform Circular Motion

When r is constant but ω varies (angular acceleration $\alpha = \dot{\omega} \neq 0$):

$$\boldsymbol{a} = \underbrace{-r\omega^2\,\hat{\boldsymbol{r}}}_{\text{centripetal}} + \underbrace{r\alpha\,\hat{\boldsymbol{\theta}}}_{\text{tangential}}\,. \tag{4.12}$$

The total acceleration has magnitude $a = \sqrt{a_c^2 + a_t^2}$ and points neither toward the center nor along the tangent, but at an angle $\phi = \arctan(a_t/a_c)$ from the radial direction.

4.4 Relative Velocity

The velocity of an object depends on the reference frame in which it is measured. If frame B moves with velocity $\boldsymbol{v}_{BA}$ relative to frame A, and a particle has velocity $\boldsymbol{v}_{PB}$ relative to frame B, then its velocity relative to frame A is:

$$\boldsymbol{v}_{PA} = \boldsymbol{v}_{PB} + \boldsymbol{v}_{BA}. \tag{4.13}$$

This is simply vector addition. If the relative velocity between the frames is constant ($\boldsymbol{v}_{BA} = \text{const}$), then the accelerations measured in both frames are the same: $\boldsymbol{a}_{PA} = \boldsymbol{a}_{PB}$.

Example 4.4 (River crossing). A boat can travel at 5.0 m/s in still water. A river is 80 m wide and flows east at 3.0 m/s. The boat aims due north. (a) Find the boat's velocity relative to the ground. (b) How long does the crossing take? (c) How far downstream does the boat land?

Solution. Let north $= \hat{\boldsymbol{\jmath}}$, east $= \hat{\boldsymbol{\imath}}$. The boat's velocity relative to the water is $\boldsymbol{v}_{BW} = 5.0\hat{\boldsymbol{\jmath}}$ m/s. The water's velocity relative to the ground is $\boldsymbol{v}_{WG} = 3.0\hat{\boldsymbol{\imath}}$ m/s. By (4.13):

$$\boldsymbol{v}_{BG} = \boldsymbol{v}_{BW} + \boldsymbol{v}_{WG} = 3.0\hat{\boldsymbol{\imath}} + 5.0\hat{\boldsymbol{\jmath}} \text{ m/s}.$$

(a) Speed: $|\boldsymbol{v}_{BG}| = \sqrt{9+25} = 5.83$ m/s at angle $\arctan(3/5) = 31.0°$ east of north.

(b) The northward component is 5.0 m/s, so crossing time $= 80/5.0 = 16$ s.

(c) Downstream drift $= 3.0 \times 16 = 48$ m.

To land directly across, the boat must aim upstream at angle $\alpha = \arcsin(3/5) = 36.9°$ west of north, giving a crossing speed of $\sqrt{25-9} = 4.0$ m/s and crossing time $80/4.0 = 20$ s.

Example 4.5 (Centrifuge). A centrifuge of radius $R = 0.10$ m spins at $n = 10\,000$ rpm. Find the centripetal acceleration and express it in multiples of g.

Solution. Angular velocity: $\omega = 2\pi n/60 = 2\pi(10000)/60 = 1047$ rad/s. Centripetal acceleration: $a_c = R\omega^2 = 0.10 \times (1047)^2 = 1.10 \times 10^5$ m/s^2. In multiples of g: $a_c/g = 1.10 \times 10^5/9.8 \approx 11{,}200\,g$. This enormous acceleration is why centrifuges are effective at separating substances of different densities.

Problems

Problem 4.1 ⋆

A ball is thrown horizontally at 15 m/s from an 80 m cliff. Find (a) time in air, (b) horizontal distance, (c) impact speed. Use $g = 10\,\text{m/s}^2$.

Problem 4.2 ⋆⋆

A projectile is launched at $v_0 = 40\,\text{m/s}$, $\theta = 60°$ from ground level. Use $g = 10\,\text{m/s}^2$. (a) Find time of flight, max height, and range. (b) What other angle gives the same range? (c) Compare the two trajectories.

Problem 4.3 ⋆⋆⋆

A particle moves in a circle of radius R with $\theta(t) = \omega_0 t + \frac{1}{2}\alpha t^2$. (a) Find $\dot{\theta}$ and $\ddot{\theta}$. (b) Find the centripetal and tangential accelerations. (c) When is $|a_c| = |a_t|$?

Problem 4.4 ⋆⋆⋆⋆

A projectile is launched from a cliff of height h at angle θ *below* horizontal with speed v_0. (a) Find the range R and impact speed v_f. (b) Show that v_f is independent of θ. (c) Find the angle maximizing the range.

Problem 4.5 ⋆⋆⋆⋆⋆

A particle has $\boldsymbol{r}(t) = A\cos(\omega t)\hat{\boldsymbol{\imath}} + B\sin(\omega t)\hat{\boldsymbol{\jmath}}$ with $A > B > 0$. (a) Show the path is an ellipse. (b) Find $\boldsymbol{v}$ and $\boldsymbol{a}$. (c) Show $\boldsymbol{a} = -\omega^2\boldsymbol{r}$. (d) Find where the speed is maximum and minimum.

Problem 4.6 ⋆⋆⋆

A centrifuge of radius $R = 0.15\,\text{m}$ spins at 12 000 rpm. Express the centripetal acceleration in terms of g. If a sample of mass $m = 0.010\,\text{kg}$ sits at the rim, what is the net inward force required?

Problem 4.7 ⋆⋆⋆⋆

A bead slides along a frictionless wire bent into a spiral $r(\theta) = a\theta$ for $\theta \geq 0$ and $a > 0$. If the bead moves so that $\dot{\theta} = \omega_0$ (constant), find the velocity and acceleration of the bead as functions of θ.

Problem 4.8 ⋆⋆⋆

A cannon on a cliff of height $h = 100\,\text{m}$ fires a shell at angle $\theta = 30°$ above horizontal with $v_0 = 50\,\text{m/s}$. (a) Find the range. (b) Find the impact speed. (c) Find the impact angle. ($g = 10\,\text{m/s}^2$.)

Problem 4.9 ⋆⋆⋆

A particle moves in a circle of radius $R = 2.0\,\text{m}$ with angular velocity $\omega(t) = 3t^2$ rad/s. At $t = 1\,\text{s}$, find (a) the tangential speed, (b) the centripetal acceleration, (c) the tangential acceleration, (d) the magnitude of the total acceleration and the angle it makes with the radius.

Problem 4.10 ⋆⋆⋆⋆

A projectile is launched at speed v_0 from the origin. Show that the **envelope** of all possible trajectories (as θ varies) is the parabola $y = v_0^2/(2g) - gx^2/(2v_0^2)$, and that no projectile can reach a point above this curve.

Chapter 5

Newton's Laws of Motion

Isaac Newton's three laws, first published in the *Philosophiae Naturalis Principia Mathematica* (1687), form the foundation of classical mechanics. They describe the relationship between forces and motion, and they remain valid for all macroscopic objects moving at speeds much less than the speed of light. In this chapter we state the laws, develop the vocabulary of forces, and build the problem-solving machinery that the rest of the course depends on.

5.1 Newton's First Law and Inertial Frames

Theorem 5.1: Newton's First Law (Law of Inertia)

An object at rest remains at rest, and an object in uniform motion continues in uniform motion, unless acted upon by a net external force.

Historical Context

The First Law was revolutionary because it overturned 2000 years of Aristotelian physics. Aristotle taught that force is the *cause of motion*: a moving object requires a continuous push to keep moving, and when the push stops, the object stops. This seems intuitive (a book pushed across a table does stop) but Galileo realized that the book stops because of *friction*, an additional force opposing the motion. In the idealized limit of no friction, Galileo argued, the book would continue moving forever. Newton crystallized this insight into a precise law.

Inertial Reference Frames

The First Law does more than describe the motion of force-free objects; it *defines* the class of reference frames in which the laws of mechanics hold in their standard form. A reference frame is **inertial** if Newton's First Law is valid in it: that is, if a free particle (one subject to no forces) moves in a straight line at constant speed.

Any frame moving at constant velocity relative to an inertial frame is also inertial. A frame that accelerates relative to an inertial frame is **non-inertial**: in such a frame, free particles appear to accelerate even though no physical force acts on them. This is why passengers in a braking bus lurch forward, and why a ball on the dashboard of a turning car appears to slide sideways.

Common Mistake 5.1: Is the First Law Just a Special Case of the Second?

A common misconception is that the First Law is merely $\boldsymbol{F}_{\text{net}} = m\boldsymbol{a}$ with $\boldsymbol{F}_{\text{net}} = \mathbf{0}$, making it redundant. In fact, the First Law plays an independent logical role: it defines the existence of inertial frames. The Second Law then tells us how objects accelerate *within* those frames. Without the First Law, the Second Law would be circular: it would presuppose the very frames it needs to make sense.

5.2 Newton's Second Law

Theorem 5.2: Newton's Second Law

The net force on an object equals the rate of change of its momentum:

$$\boldsymbol{F}_{\text{net}} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t}. \tag{5.1}$$

For constant mass, this reduces to $\boldsymbol{F}_{\text{net}} = m\boldsymbol{a}$. In components: $\sum F_x = ma_x$, $\sum F_y = ma_y$, $\sum F_z = ma_z$.

The general form $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$ is more fundamental because it applies even when the mass changes (as in rocket propulsion). When we reach the chapter on momentum, we will use this general form extensively.

The Superposition Principle

A crucial feature implicit in the Second Law is the **superposition principle**: the net force on an object is the vector sum of all individual forces acting on it,

$$\boldsymbol{F}_{\text{net}} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots + \boldsymbol{F}_N.$$

Each force contributes independently; the presence of one force does not modify the effect of another. This is an empirical fact, not a logical necessity, and it allows us to analyze complicated situations by considering one force at a time.

Mass as a Measure of Inertia

Mass quantifies an object's resistance to acceleration. If the same net force $\boldsymbol{F}$ applied to two different objects produces accelerations $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, then their mass ratio is

$$\frac{m_1}{m_2} = \frac{a_2}{a_1}.$$

This provides an *operational definition* of mass: by comparing accelerations under the same force, we can assign masses to all objects relative to a chosen standard. Mass is an intrinsic property of an object: it does not change with location, speed (at least at non-relativistic speeds), or the coordinate system used.

Key Point 5.1: Dimensional Consistency

The Second Law requires force, mass, and acceleration to have consistent units. In SI: $[\text{force}] = \mathrm{kg\,m/s^2} \equiv \mathrm{N}$. Dimensional analysis is a powerful tool for checking results: any expression for a force must have dimensions of $[\text{mass}][\text{length}][\text{time}]^{-2}$.

5.3 Newton's Third Law

Theorem 5.3: Newton's Third Law

If body A exerts a force on body B, then body B exerts an equal and opposite force on body A:

$$\boldsymbol{F}_{A\to B} = -\boldsymbol{F}_{B\to A}. \tag{5.2}$$

These forces act on *different* objects and cannot cancel each other.

Action–reaction pairs are *always* of the same type: if A pushes B by contact, then B pushes A by contact; if Earth pulls the ball by gravity, then the ball pulls Earth by gravity. They are always equal in magnitude, opposite in direction, and simultaneous.

Identifying Action–Reaction Pairs

A reliable method: for any force "A pushes B," the reaction is "B pushes A." Simply swap the two objects and reverse the direction. If you cannot name two distinct objects, you have not correctly identified the force.

Example. A book rests on a table. The forces on the book are (1) gravity (Earth pulls book down) and (2) the normal force (table pushes book up). These are *not* an action–reaction pair: they act on the *same* object. The reaction to (1) is "book pulls Earth up" (a gravitational force on Earth). The reaction to (2) is "book pushes table down" (a contact force on the table).

The Horse-and-Cart Paradox

This is the classic Third-Law puzzle: a horse pulls a cart, but by the Third Law, the cart pulls the horse backward with an equal force. If the forces are equal and opposite, how can anything move?

The resolution comes from carefully applying the Second Law to each object separately:

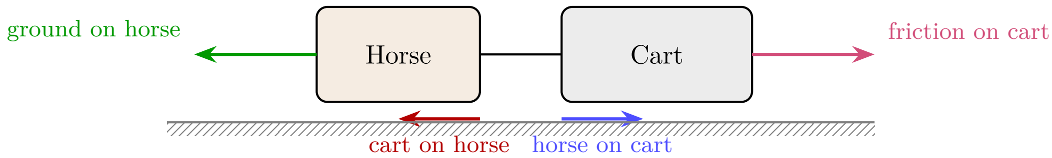


Figure 5.3.1: The horse-and-cart system. The Third-Law pair (horse on cart / cart on horse) are equal and opposite but act on *different* objects. The horse accelerates if the ground pushes it forward more than the cart pulls it back.

For the horse: The ground pushes the horse forward (reaction to the horse's hooves pushing backward), and the cart pulls the horse backward. If the ground force exceeds the backward pull, the horse has a net forward force and accelerates.

For the cart: The horse pulls the cart forward, and friction (or road resistance) pushes the cart backward. If the horse's pull exceeds the friction, the cart accelerates forward.

The Third-Law pair (horse on cart / cart on horse) are equal and opposite, but they act on *different* objects and therefore do not cancel. Motion is determined by the *net force on each individual object*, not by comparing forces across objects.

Common Mistake 5.2: Forces on Different Objects Don't Cancel

Action–reaction pairs *never* cancel because they act on different objects. Only forces acting on the *same* object can cancel. When you draw a free-body diagram, you include only the forces on one object: the reaction forces appear on the FBD of the *other* object.

5.4 Common Forces in Mechanics

5.4.1 Weight (Gravitational Force)

Near Earth's surface, the gravitational force on an object of mass m is

$$\boldsymbol{W} = m\boldsymbol{g},$$

directed vertically downward with magnitude $W = mg$. Weight is a *force*, measured in newtons; mass is a property of the object, measured in kilograms. An astronaut has the same mass on the Moon as on Earth, but weighs only one-sixth as much because $g_{\text{Moon}} \approx g/6$.

5.4.2 The Normal Force

When an object presses against a surface, the surface deforms microscopically and pushes back perpendicular to itself. This perpendicular contact force is the **normal force $\boldsymbol{N}$**. It is a *constraint force*: it adjusts its magnitude to whatever value is needed to prevent the object from passing through the surface.

Common Mistake 5.3: $N \neq mg$ in General

The normal force equals mg *only* for an object resting on a horizontal surface with no other vertical forces. On an incline, $N = mg\cos\theta$. In an accelerating elevator, $N = m(g + a)$. In circular motion at the top of a hill, $N = mg - mv^2/R$. Always derive N from Newton's second law; never assume $N = mg$.

Example 5.A (Apparent weight in an elevator). A person of mass m stands on a bathroom scale in an elevator. The scale reads the normal force N. Applying $\sum F_y = ma_y$ with upward positive:

$$N - mg = ma \quad\Longrightarrow\quad N = m(g + a).$$

When the elevator accelerates upward ($a > 0$), the person feels heavier ($N > mg$). When it accelerates downward ($a < 0$), the person feels lighter ($N < mg$). In free fall ($a = -g$), $N = 0$: apparent weightlessness.

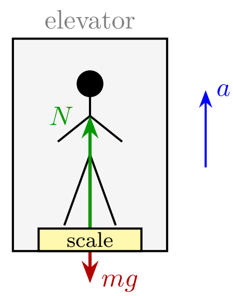


Figure 5.4.1: Free-body diagram for a person in an accelerating elevator. The scale reads $N = m(g + a)$.

5.4.3 Tension

The force transmitted through a taut string, rope, or cable is called **tension**. For an *ideal string* (massless and inextensible), the tension is the same throughout; this follows from applying $F = ma$ to a massless element of string, which would have infinite acceleration unless the net force on it is zero.

For a *massive string* of total mass m_s and length L being pulled by force F at one end (with the other end attached to a block of mass M), the tension varies along the string. At distance x from the block, the portion of string beyond x has mass $m_s(L - x)/L$. Applying Newton's second law:

$$T(x) = (M + m_s x/L)\, a, \qquad \text{where} \quad a = \frac{F}{M + m_s}.$$

In the limit $m_s \to 0$, this reduces to $T = Ma = \text{const}$ throughout, recovering the ideal-string result.

5.4.4 Friction

Static friction prevents the onset of relative motion between two surfaces in contact:

$$f_s \le \mu_s N. \tag{5.3}$$

The inequality is essential: static friction adjusts its magnitude from zero up to the maximum value $\mu_s N$ to match whatever is needed to prevent sliding. Only at the threshold of slipping does $f_s = \mu_s N$.

Kinetic friction opposes the direction of sliding:

$$f_k = \mu_k N, \tag{5.4}$$

with μ_k typically less than μ_s. The kinetic friction force is approximately constant and independent of speed (though this is an idealization; at very high or very low speeds, the model breaks down).

Key Point 5.2: Limitations of the Friction Model

The Amontons–Coulomb friction model ($f \le \mu N$, f independent of contact area) is an empirical approximation that works well for dry, hard surfaces at moderate loads. It fails for very smooth surfaces, lubricated contacts, rubber on glass, and other situations. Nevertheless, it captures the essential physics for nearly all problems in introductory mechanics.

Critical angle for sliding. A block on an incline begins to slide when the component of gravity along the surface exceeds maximum static friction: $mg \sin\theta > \mu_s mg \cos\theta$, giving $\tan\theta_{\max} = \mu_s$. This provides a simple experimental method for measuring μ_s.

5.4.5 Drag Forces

At low speeds (small Reynolds number), the drag on a sphere in a viscous fluid is linear in velocity: $f = bv$ (Stokes drag). At high speeds (large Reynolds number), the drag is quadratic: $f = Dv^2$ (turbulent drag), where $D = \frac{1}{2}C_D\rho A$ depends on the drag coefficient C_D, fluid density ρ, and cross-sectional area A.

5.4.6 Spring Force (Hooke's Law)

$F = -kx$ is a restoring force proportional to displacement from equilibrium, where k is the spring constant (N/m). This linear approximation is valid for sufficiently small deformations and is the foundation of simple harmonic motion (Chapter 18).

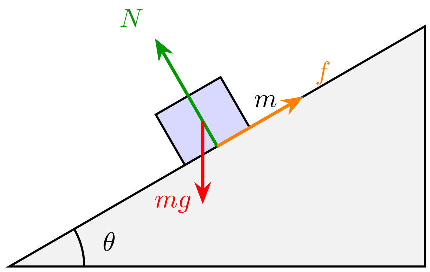


Figure 5.4.2: Free-body diagram for a block on an inclined plane, showing the weight mg, normal force N, and friction f

5.5 Problem-Solving Strategy

Strategy 5.1: Applying Newton's Laws

1. **Identify the system:** What object(s) are you analyzing?
2. **Draw a free-body diagram (FBD):** Show *all* forces acting *on* the object. Do not include forces the object exerts on other things.
3. **Choose a coordinate system:** Align one axis along the acceleration when possible.
4. **Apply $\sum F_x = ma_x$ and $\sum F_y = ma_y$:** Write one equation per component.
5. **Solve algebraically**, then substitute numbers last.
6. **Check:** Do signs, units, and limiting cases make sense?

5.6 Applications of Newton's Laws

5.6.1 Inclined Planes

For a block of mass m on an incline at angle θ:

- Frictionless: $a = g\sin\theta$, $N = mg\cos\theta$.
- With kinetic friction: $a = g(\sin\theta - \mu_k\cos\theta)$.
- Critical angle for static slip: $\tan\theta_{\max} = \mu_s$.

5.6.2 Circular Motion Applications

The centripetal acceleration $a_c = v^2/r$ must be provided by the net radial force. Newton's second law in the radial direction gives $\sum F_r = mv^2/r$.

Car over a hill. At the crest of a hill with radius of curvature R, gravity points inward (toward the center of the circular path) and the normal force points outward:

$$mg - N = \frac{mv^2}{R} \quad \Longrightarrow \quad N = m\left(g - \frac{v^2}{R}\right).$$

The car loses contact when $N = 0$, giving the maximum speed $v_{\max} = \sqrt{gR}$.

Banked curve (no friction). On a frictionless banked curve at angle θ, the horizontal component of the normal force provides the centripetal force:

$$N\sin\theta = \frac{mv^2}{R}, \qquad N\cos\theta = mg \quad \Longrightarrow \quad \boxed{v = \sqrt{gR\tan\theta}.}$$

This is the *design speed* of the curve: the unique speed at which no friction is needed.

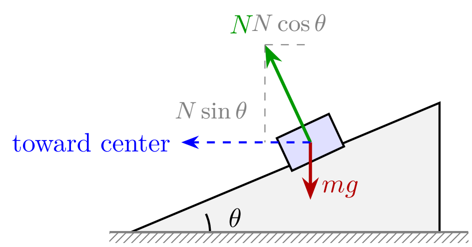


Figure 5.6.1: Free-body diagram for a car on a frictionless banked curve. The horizontal component of N provides the centripetal force.

5.6.3 Drag Forces and Terminal Velocity

For a sphere falling through a viscous fluid with drag $\boldsymbol{f} = -b\boldsymbol{v}$, Newton's second law gives:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - bv.$$

This is a first-order linear ODE with solution (see Appendix A.5):

$$v(t) = v_T\left(1 - e^{-t/\tau}\right), \qquad v_T = \frac{mg}{b}, \quad \tau = \frac{m}{b}. \tag{5.5}$$

At long times, $v \to v_T$ (terminal velocity). For small t, $v \approx gt$ (free fall), recovering the familiar result.

5.6.4 Coupled Systems and Constraints

When multiple objects are connected (e.g., by strings over pulleys), the key insight is that the connections impose **constraints** on the motion. An inextensible string forces both ends to have the same magnitude of acceleration; a rigid contact forces two surfaces to share a common acceleration component.

Strategy for coupled systems:

1. Draw a separate FBD for each object.

2. Identify the constraint relating the accelerations (e.g., $a_1 = a_2$ for an inextensible string).
3. Write $\sum F = ma$ for each object.
4. Solve the resulting system of equations.

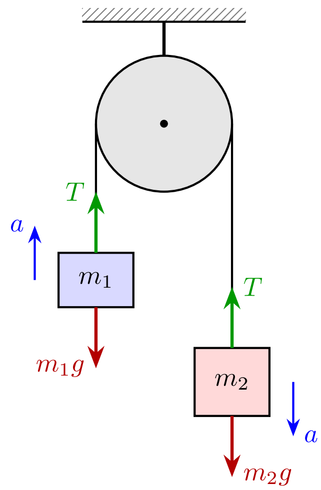


Figure 5.6.2: The Atwood machine: two masses connected by a massless string over a frictionless, massless pulley. The constraint is that both masses have the same magnitude of acceleration a.

5.6.5 Worked Examples

Example 5.1 (Two blocks and a pulley). A block of mass m_1 sits on a frictionless table, connected by a massless string over a massless, frictionless pulley to a hanging block of mass m_2. Find the acceleration and tension.

Solution. For m_1 (horizontal): $T = m_1 a$. For m_2 (vertical, downward positive): $m_2 g - T = m_2 a$. Adding: $m_2 g = (m_1 + m_2)a$, so

$$a = \frac{m_2 g}{m_1 + m_2}, \qquad T = \frac{m_1 m_2 g}{m_1 + m_2}.$$

Checks: If $m_2 = 0$, $a = 0$ and $T = 0$ (nothing pulls). If $m_1 = 0$, $a = g$ and $T = 0$ (free fall). If $m_1 \to \infty$, $a \to 0$ and $T \to m_2 g$ (the hanging block is essentially held in place).

Example 5.2 (Block on a block with friction). A block of mass m_1 sits atop a block of mass m_2 on a frictionless floor. A horizontal force F is applied to m_2. The coefficient of static friction between the blocks is μ_s. Find the maximum F such that the blocks move together.

Solution. When they move together, $a = F/(m_1 + m_2)$. The friction on m_1 (from m_2) provides its acceleration: $f = m_1 a = m_1 F/(m_1 + m_2)$. For no slipping, $f \leq \mu_s m_1 g$:

$$\frac{m_1 F}{m_1 + m_2} \leq \mu_s m_1 g \quad \Longrightarrow \quad F_{\max} = \mu_s (m_1 + m_2) g.$$

Example 5.3 (Velocity-dependent drag). A sky diver of mass m falls from rest through air with quadratic drag $F_d = Dv^2$. Find the terminal velocity and describe the motion qualitatively.

Solution. Newton's second law: $m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - Dv^2$. At terminal velocity, $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$: $mg = Dv_T^2$, giving $v_T = \sqrt{mg/D}$. For early times, $v \ll v_T$, so $a \approx g$ (free fall). As v increases, the drag increases and the acceleration decreases. The speed asymptotically approaches v_T. This ODE can be solved by separation:

$$\frac{\mathrm{d}v}{g-(D/m)v^2}=\mathrm{d}t \quad \Longrightarrow \quad v(t)=v_T\tanh\left(\frac{gt}{v_T}\right).$$

5.7 Relative Motion and Reference Frames

We introduced the relative velocity formula $\boldsymbol{v}_{PA}=\boldsymbol{v}_{PB}+\boldsymbol{v}_{BA}$ in Chapter 4 as a kinematic tool for problems involving multiple moving objects. Here we revisit it from the perspective of Newton's laws, because the choice of reference frame has far-reaching implications for the validity of $\boldsymbol{F}=m\boldsymbol{a}$.

Galilean Velocity Transformation

If frame S' moves at constant velocity $\boldsymbol{V}$ relative to frame S, the position and velocity of a particle P are related by:

$$\boldsymbol{r}_{PS}=\boldsymbol{r}_{PS'}+\boldsymbol{V}t, \qquad \boldsymbol{v}_{PS}=\boldsymbol{v}_{PS'}+\boldsymbol{V}. \tag{5.6}$$

Since $\boldsymbol{V}$ is constant, differentiating the velocity relation gives:

$$\boldsymbol{a}_{PS}=\boldsymbol{a}_{PS'}. \tag{5.7}$$

The acceleration is the same in both frames. This means that if $\boldsymbol{F}=m\boldsymbol{a}$ holds in frame S, it also holds in S' with the same force $\boldsymbol{F}$ and the same acceleration.

The Principle of Galilean Relativity

This leads to a striking conclusion: *Newton's laws take the same form in all inertial reference frames.* No mechanical experiment performed inside a closed laboratory can determine whether the lab is "at rest" or moving at constant velocity. This is the **principle of Galilean relativity**.

The principle explains why you can toss a ball on an airplane flying at constant velocity and catch it exactly as you would on the ground: the laws of motion are identical in both frames. It also means there is no such thing as "absolute rest": only relative motion is physically meaningful.

Example 5.4 (Ball dropped in a moving train). A person on a train moving at constant velocity V drops a ball from rest (relative to the train) at height h. Describe the motion in (a) the train frame and (b) the ground frame. Verify that Newton's second law holds in both.

Solution. (a) In the train frame: the ball starts at rest and falls straight down with $a=g$. It hits the floor directly below the release point after time $t=\sqrt{2h/g}$. Newton's second law: $F=mg$ (downward), $a=g$. ✓

(b) In the ground frame: the ball has initial horizontal velocity V and zero vertical velocity. It follows a parabolic trajectory: $x(t)=Vt$, $y(t)=h-\frac{1}{2}gt^2$. The only force is gravity (mg, downward), so $a_x=0$ and $a_y=-g$. Newton's second law: $\boldsymbol{F}=m\boldsymbol{a}$ with $\boldsymbol{F}=-mg\,\hat{\boldsymbol{\jmath}}$ and $\boldsymbol{a}=-g\,\hat{\boldsymbol{\jmath}}$. ✓

The ball's *trajectory* looks different in the two frames (straight line vs. parabola), but the *force* and *acceleration* are identical. This is Galilean relativity in action.

Problems

Problem 5.1 ⋆

A 5.0 kg box is pushed with 30 N on a floor with $\mu_k = 0.20$. Use $g = 10\,\mathrm{m/s^2}$. (a) Find the friction force. (b) Find the acceleration. (c) Find the speed after 4.0 s from rest.

Problem 5.2 ⋆⋆

Blocks $m_1 = 3.0\,\mathrm{kg}$ and $m_2 = 5.0\,\mathrm{kg}$ are connected by a string on a frictionless surface. Force $F = 24\,\mathrm{N}$ is applied to m_2. (a) Find the acceleration. (b) Find the string tension. (c) Maximum F if the string can hold 15 N?

Problem 5.3 ⋆⋆⋆

A block of mass m rests on an incline at angle θ with friction coefficients μ_s, μ_k. (a) Find $\theta_{\max}$ before sliding. (b) For $\theta < \theta_{\max}$, find the friction force. (c) For $\theta > \theta_{\max}$, find the acceleration.

Problem 5.4 ⋆⋆⋆⋆

A conical pendulum: mass m, string length L, circular orbit radius $R < L$. (a) Find tension T in terms of m, g, L, R. (b) Find speed v. (c) Find period τ. (d) Show $\tau \to 2\pi\sqrt{L/g}$ as $R \to 0$.

Problem 5.5 ⋆⋆⋆⋆⋆

A sphere falls through a viscous fluid with drag $\boldsymbol{f} = -b\boldsymbol{v}$. (a) Write the differential equation for $v(t)$. (b) Solve with $v(0) = 0$. (c) Find terminal velocity and position $y(t)$. (d) Show $y \approx \frac{1}{2}gt^2$ for small t.

Problem 5.6 ⋆⋆⋆

A car of mass m travels over a hill whose crest has radius of curvature R. (a) Draw a free-body diagram at the crest. (b) Find the normal force at the crest as a function of speed. (c) At what speed does the car lose contact with the road?

Problem 5.7 ⋆⋆⋆⋆

Two blocks of masses m_1 and m_2 ($m_2 > m_1$) are connected by a massless string over a frictionless, massless pulley (a simple Atwood machine). (a) Show that $a = (m_2 - m_1)g/(m_1 + m_2)$. (b) Find the tension T. (c) Show that in the limit $m_1 = m_2$, $a = 0$ and $T = mg$. (d) Show that in the limit $m_1 \ll m_2$, $a \to g$ and $T \to 2m_1 g$.

Problem 5.8 ⋆⋆⋆

A block of mass m is pushed against a vertical wall by a horizontal force F. The coefficient of static friction between block and wall is μ_s. Find the minimum F to prevent the block from sliding down.

Problem 5.9 ⋆⋆⋆⋆

A bead of mass m slides without friction on a circular wire of radius R in a vertical plane. (a) Using Newton's second law in polar coordinates, find the normal force exerted by the wire at angle θ from the bottom, given that the bead is released from rest at the top. (b) At what angle does the bead leave the wire? (Compare with the "ball sliding off a dome" problem.)

Problem 5.10 ⋆⋆⋆

Three blocks of masses m_1, m_2, m_3 are connected by strings and pulled along a frictionless surface by force F applied to m_3. Find the acceleration and the tension in each string.

Chapter 6

Work and Power

Energy is one of the most fundamental and unifying concepts in all of physics. In this chapter we introduce the first of the energy concepts: **work**, the mechanism by which a force transfers energy to or from an object. We also introduce **power**, the rate at which this transfer occurs. Together with the kinetic energy and the work-energy theorem (Chapter 7) and potential energy (Chapter 8), these ideas will give us a powerful alternative to Newton's second law for solving problems in mechanics.

6.1 Work Done by a Constant Force

Definition 6.1: Work by a Constant Force

When a constant force $\boldsymbol{F}$ acts on an object that undergoes displacement $\boldsymbol{d}$, the work done by the force is

$$W = \boldsymbol{F} \cdot \boldsymbol{d} = Fd\cos\theta, \tag{6.1}$$

where θ is the angle between $\boldsymbol{F}$ and $\boldsymbol{d}$. Work is a scalar with SI unit joules ($\mathrm{J} = \mathrm{N\,m} = \mathrm{kg\,m^2/s^2}$).

The geometry is shown in Figure 6.1.1. The factor $\cos\theta$ projects the force onto the direction of motion: only the component of force *along* the displacement does work. The perpendicular component changes the direction of motion but does not speed the object up or slow it down.

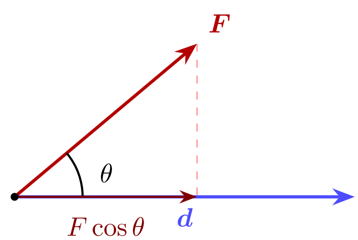


Figure 6.1.1: Geometry of work by a constant force. Only the component of $\boldsymbol{F}$ along the displacement $\boldsymbol{d}$—namely $F\cos\theta$—contributes to the work.

Key Point 6.1: Work is a Scalar

Although work is defined using two vectors ($\boldsymbol{F}$ and $\boldsymbol{d}$), their dot product is a scalar (a single number with no direction). Work can be positive (energy added), negative (energy removed), or zero (no energy transfer). This is fundamentally different from force and displacement, which are vectors.

The Sign of Work

The sign of $W = Fd\cos\theta$ has a direct physical interpretation:

- $\theta < 90°$: $W > 0$ (the force has a component along the displacement; the force *adds* energy to the object).
- $\theta = 90°$: $W = 0$ (force perpendicular to motion does no work; e.g., the centripetal force in circular motion, or the normal force on a frictionless surface).
- $\theta > 90°$: $W < 0$ (the force has a component opposing the displacement; the force *removes* energy from the object, e.g., kinetic friction).

When Does the Normal Force Do Work?

Since the normal force is always perpendicular to the contact surface, students often conclude that it never does work. This is correct when the surface itself is stationary: for example, a block sliding on a fixed ramp. In that case, the displacement of the block is along the surface, perpendicular to $\boldsymbol{N}$, so $W_N = 0$.

However, when the surface *moves*, the normal force can do nonzero work. Consider a person standing in an elevator that accelerates upward. The floor exerts a normal force N upward on the person, and the person's displacement is upward (the person moves with the elevator). The angle between $\boldsymbol{N}$ and $\boldsymbol{d}$ is zero, so $W_N = Nd > 0$—the normal force does positive work. It is this work that increases the person's kinetic and potential energy.

Key Point 6.2: Work Requires Displacement of the Point of Application

The work done by a force depends on the *displacement of the point where the force is applied*, not the displacement of the object's center of mass. For a stationary surface, the contact point doesn't move, so $W_N = 0$. For a moving surface (elevator, conveyor belt), the contact point moves with the surface, and $W_N \neq 0$.

Work is Frame-Dependent

Because the displacement $\boldsymbol{d}$ of an object depends on the reference frame of the observer, the work done by a force is also frame-dependent. A ball dropped inside a moving train undergoes zero horizontal displacement in the train's frame but nonzero horizontal displacement in the ground frame. The work done by gravity is therefore different in the two frames.

This may seem alarming, but it is not a contradiction: the work-energy theorem $W_{\text{net}} = \Delta K$ holds in every inertial frame, and both W_{net} and ΔK change in exactly the same way under a change of frame. The physics is self-consistent in any inertial frame.

Work in Component Form

When force and displacement are given in Cartesian components, the dot product gives:

$$W = F_x d_x + F_y d_y + F_z d_z. \tag{6.2}$$

This is often the most practical route to computing work when $\boldsymbol{F}$ and $\boldsymbol{d}$ are specified in terms of $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$.

Key Point 6.3: Work is Done by Individual Forces

Each force acting on an object does its own work, independently of the other forces. The *net work* is the sum of work done by all forces: $W_{\text{net}} = \sum_i W_i$. Equivalently, $W_{\text{net}} = \boldsymbol{F}_{\text{net}} \cdot \boldsymbol{d}$. Both methods give the same result, but tracking work by each force individually is essential for the energy methods we develop in later chapters.

Worked Examples

Example 6.1 (Work by gravity along different paths). A 2 kg block moves from point A at height $h = 5$ m to point B at height 0 along three different paths: (i) straight down, (ii) down a 30° ramp, (iii) over a hill and then down. Show that the work done by gravity is the same for all three paths.

Solution. In all cases, $W_g = -mg\Delta y = -mg(0 - h) = mgh = 2(9.8)(5) = 98$ J. Gravity is a conservative force; its work depends only on the change in height, not the path. This is the fundamental property that allows us to define gravitational potential energy.

Example 6.2 (Pulling a sled at an angle). A child pulls a sled of mass m a horizontal distance d along level ground by a rope that makes angle ϕ above the horizontal, with constant tension T. The coefficient of kinetic friction is μ_k. Find the work done by each force.

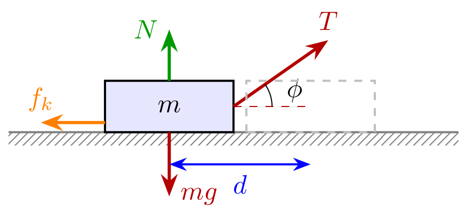


Figure 6.1.2: A sled pulled at angle ϕ above horizontal. Only the horizontal component of T does work along the displacement.

Solution. The displacement is horizontal: $\boldsymbol{d} = d\,\hat{\boldsymbol{\imath}}$.

Tension: $W_T = Td\cos\phi$. Only the horizontal component $T\cos\phi$ contributes.

Gravity: $W_g = 0$ (gravity is perpendicular to the horizontal displacement).

Normal force: $W_N = 0$ (perpendicular to displacement).

Friction: $W_f = -f_k d = -\mu_k N d$. To find N, apply Newton's second law vertically ($a_y = 0$): $N + T\sin\phi - mg = 0$, so $N = mg - T\sin\phi$. Therefore $W_f = -\mu_k(mg - T\sin\phi)\,d$.

Note that the upward pull $T\sin\phi$ reduces the normal force and hence reduces the friction; this is why pulling at an angle can be more efficient than pushing horizontally.

Net Work: Two Equivalent Methods

Given the work done by each individual force, the net work can be computed in two equivalent ways:

Method 1: Sum individual works. Compute the work done by each force separately and add them: $W_{\text{net}} = W_1 + W_2 + \cdots$.

Method 2: Use the net force. First find $\boldsymbol{F}_{\text{net}} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots$, then compute $W_{\text{net}} = \boldsymbol{F}_{\text{net}} \cdot \boldsymbol{d}$.

Both methods always give the same result. Method 1 is more informative because it reveals how much energy each force contributes; Method 2 can be faster when only the net result is needed.

Example 6.C (Verifying equivalence). Returning to the sled in Example 6.2, find the net work both ways. Take $m = 10$ kg, $T = 50$ N, $\phi = 30°$, $\mu_k = 0.20$, $d = 8.0$ m, $g = 10\,\text{m/s}^2$.

Method 1:

$$W_T = Td\cos\phi = 50(8)\cos 30° = 346\,\text{J}, \qquad W_g = 0, \qquad W_N = 0.$$

Normal force: $N = mg - T\sin\phi = 100 - 25 = 75\,\text{N}$. Friction: $W_f = -\mu_k N d = -0.20(75)(8) = -120\,\text{J}$.

$$W_{\text{net}} = 346 + 0 + 0 - 120 = 226\,\text{J}.$$

Method 2: The net horizontal force is $F_{\text{net}} = T\cos\phi - \mu_k N = 50\cos 30° - 15 = 43.3 - 15 = 28.3\,\text{N}$. (The vertical net force is zero since the sled doesn't accelerate vertically.)

$$W_{\text{net}} = F_{\text{net}} \cdot d = 28.3 \times 8 = 226\,\text{J}. \quad \checkmark$$

6.2 Work Done by a Variable Force

When the force varies along the path, we cannot simply multiply force by displacement. Instead, we divide the path into infinitesimal segments $\mathrm{d}\boldsymbol{r}$, compute the infinitesimal work $\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ on each segment, and sum (integrate) over the entire path:

$$W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_C (F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z). \tag{6.3}$$

In one dimension, this reduces to the familiar definite integral:

$$W = \int_{x_i}^{x_f} F(x)\,\mathrm{d}x. \tag{6.4}$$

Graphical Interpretation

In one dimension, the work done by a force $F(x)$ equals the **area under the $F(x)$ curve** between x_i and x_f, with regions below the x-axis counting as negative area. This is a direct consequence of the integral being a signed area.

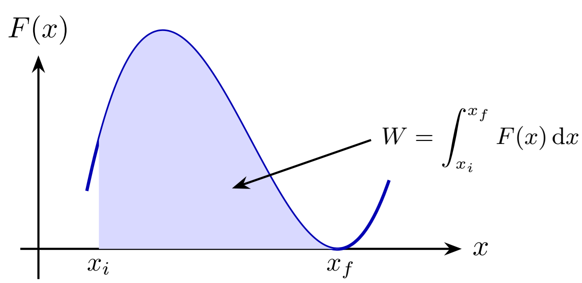


Figure 6.2.1: The work done by a variable force equals the signed area under the $F(x)$ curve.

Example 6.G (Reading work from a graph). A particle moves along the x-axis under the force shown in Figure 6.2.2. Find the work done from $x = 0$ to $x = 5d$.

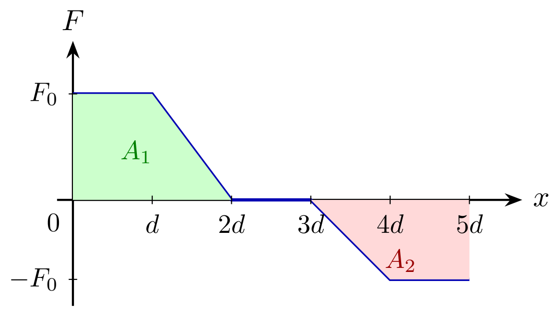


Figure 6.2.2: A piecewise-linear force. The work is the sum of the signed areas: A_1 (positive, green) and A_2 (negative, red).

Solution. We compute the area of each region:

Region A_1 ($0 \le x \le 2d$): From 0 to d, $F = F_0$ (constant), giving a rectangular area $F_0 \cdot d$. From d to $2d$, F drops linearly from F_0 to 0, giving a triangular area $\frac{1}{2}F_0 \cdot d$. Total positive area: $A_1 = F_0 d + \frac{1}{2}F_0 d = \frac{3}{2}F_0 d$.

Region from $2d$ to $3d$: $F = 0$, so the area is zero.

Region A_2 ($3d \le x \le 5d$): From $3d$ to $4d$, F drops linearly from 0 to $-F_0$ (a triangle below the axis with area $-\frac{1}{2}F_0 d$). From $4d$ to $5d$, $F = -F_0$ (constant, area $-F_0 d$). Total: $A_2 = -\frac{1}{2}F_0 d - F_0 d = -\frac{3}{2}F_0 d$.

The total work is $W = A_1 + A_2 = \frac{3}{2}F_0 d - \frac{3}{2}F_0 d = 0$. The positive work done in the first half is exactly cancelled by the negative work in the second half. Note, however, that the net displacement is $5d \neq 0$—zero work does *not* imply zero displacement.

Work Along a Curved Path in Two or Three Dimensions

In higher dimensions, the line integral (6.3) is evaluated by parameterizing the path. If the path C is described by $\boldsymbol{r}(t) = x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}}$ for t ranging from t_i to t_f, then $\mathrm{d}\boldsymbol{r} = \dot{\boldsymbol{r}}\,\mathrm{d}t = (\dot{x}\,\hat{\boldsymbol{\imath}} + \dot{y}\,\hat{\boldsymbol{\jmath}})\,\mathrm{d}t$ and

$$W = \int_{t_i}^{t_f} \boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \dot{\boldsymbol{r}}(t)\,\mathrm{d}t = \int_{t_i}^{t_f} \left[F_x\,\dot{x} + F_y\,\dot{y}\right]\mathrm{d}t. \tag{6.5}$$

Alternatively, if y can be expressed as a function of x along the path, we can eliminate the parameter and write everything in terms of x and $\mathrm{d}x$.

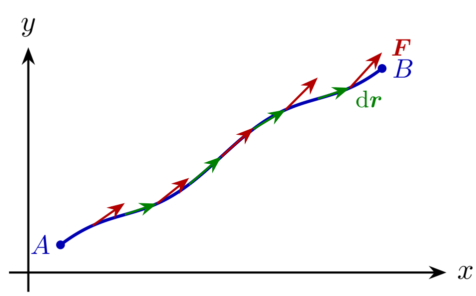


Figure 6.2.3: Computing work along a curved path. The force $\boldsymbol{F}$ (red) varies along the path from A to B. The path is divided into infinitesimal displacement vectors $\mathrm{d}\boldsymbol{r}$ (green). The work is $W = \int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$.

Example 6.A (Work along a parabolic path). A force $\boldsymbol{F} = \alpha y\,\hat{\boldsymbol{\imath}} + \beta x\,\hat{\boldsymbol{\jmath}}$ (with α and β constants having units of N/m) acts on a particle that moves along the parabola $y = cx^2$ from the origin to the point (L, cL^2). Find the work done.

Solution. Along the path, $y = cx^2$ and $\mathrm{d}y = 2cx\,\mathrm{d}x$. Substituting:

$$W = \int_C (F_x\,\mathrm{d}x + F_y\,\mathrm{d}y) = \int_0^L \left[\alpha(cx^2)\,\mathrm{d}x + \beta x(2cx\,\mathrm{d}x)\right] = \int_0^L (\alpha c + 2\beta c)\,x^2\,\mathrm{d}x = \frac{(\alpha+2\beta)cL^3}{3}.$$

Note that this result depends on the specific path ($y = cx^2$). If $\alpha \neq \beta$, different paths between the same endpoints give different amounts of work, meaning $\boldsymbol{F}$ is not conservative. If $\alpha = \beta$, one can verify that the force becomes conservative (in fact, it derives from the potential $U = -\alpha xy$).

Work Done by a Spring (Hooke's Law)

Definition 6.2: Work by a Spring Force

For a spring obeying Hooke's law ($F = -kx$, where x is measured from the natural length):

$$W_{\text{spring}} = -\int_{x_i}^{x_f} kx\,\mathrm{d}x = -\frac{1}{2}k\left[x^2\right]_{x_i}^{x_f} = \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2. \tag{6.6}$$

The derivation is worth spelling out explicitly. The spring exerts force $F = -kx$ on the object. As the object moves from x_i to x_f:

$$W_{\text{spring}} = \int_{x_i}^{x_f} (-kx)\,\mathrm{d}x = -k\int_{x_i}^{x_f} x\,\mathrm{d}x = -k\left[\frac{x^2}{2}\right]_{x_i}^{x_f} = -\frac{k}{2}(x_f^2 - x_i^2) = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2.$$

If the spring is stretched from equilibrium ($x_i = 0$) to extension x_f, the spring does negative work $W = -\frac{1}{2}kx_f^2$—the spring resists the stretching. If the spring returns from x_f to equilibrium, it does positive work $W = +\frac{1}{2}kx_f^2$—the spring gives energy back. Note that the work depends only on the initial and final positions, not on how the spring got there. This is a hallmark of a conservative force.

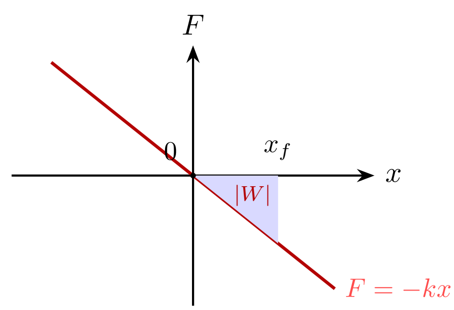


Figure 6.2.4: Hooke's law: the spring force $F = -kx$ is linear and restoring. The shaded triangle represents the magnitude of work done by the spring as it is stretched from 0 to x_f. Since the area is below the axis, the work is negative: the spring opposes the stretching.

Example 6.B (Variable force in 1D). A force $F(x) = F_0(1 - x/L)$ acts on a particle from $x = 0$ to $x = L$. This models a force that starts at F_0 and decreases linearly to zero. Find the work done.

Solution.

$$W = \int_0^L F_0\left(1 - \frac{x}{L}\right)\mathrm{d}x = F_0\left[x - \frac{x^2}{2L}\right]_0^L = F_0\left(L - \frac{L}{2}\right) = \frac{1}{2}F_0L.$$

Graphically, this is the area of a triangle with base L and height F_0, confirming the integral. Compare with a constant force F_0 over the same distance, which would do work $F_0 L$—twice as much. The factor of $\frac{1}{2}$ arises because the force diminishes along the path.

Work Done by Gravity

Definition 6.3: Work Done by Gravity

Near Earth's surface ($\boldsymbol{F}_g = -mg\hat{\boldsymbol{\jmath}}$), the work done by gravity along any path is

$$W_{\text{gravity}} = -mg(y_f - y_i) = -mg\Delta y. \tag{6.7}$$

This depends only on the change in height, not on the horizontal displacement or the path taken.

Proof of path-independence. Along any path C from point A to point B:

$$W_g = \int_C \boldsymbol{F}_g \cdot \mathrm{d}\boldsymbol{r} = \int_C (-mg\hat{\boldsymbol{\jmath}}) \cdot (\mathrm{d}x\,\hat{\boldsymbol{\imath}} + \mathrm{d}y\,\hat{\boldsymbol{\jmath}} + \mathrm{d}z\,\hat{\boldsymbol{k}}) = -mg \int_C \mathrm{d}y = -mg(y_B - y_A).$$

The $\mathrm{d}x$ and $\mathrm{d}z$ terms vanish because gravity has no x- or z-component. The integral of $\mathrm{d}y$ along *any* path from A to B equals $y_B - y_A$, regardless of the path. Therefore the work done by gravity is path-independent: it depends only on the endpoints.

Work Done by Friction

Unlike gravity and the spring force, kinetic friction is *not* a conservative force: the work it does depends on the path. For an object sliding a distance s along a surface with kinetic friction $f_k = \mu_k N$:

$$W_{\text{friction}} = -f_k \cdot s = -\mu_k N \cdot s. \tag{6.8}$$

The negative sign reflects that friction always opposes the direction of sliding. The crucial point is that s is the total *distance* traveled, not the displacement. An object that slides from A to B and back to A has zero displacement but has traveled distance $2s$, so friction does work $-2f_k s$, not zero. This path dependence is exactly what prevents us from defining a "friction potential energy."

Common Mistake 6.1: Friction and Path Dependence

If a force does nonzero work on a round trip (start and end at the same point), the force is non-conservative. Friction always does negative work proportional to the total path length, so its work on any closed path is negative, never zero. This is why mechanical energy is not conserved when friction is present: the lost energy goes into thermal energy.

6.3 Power

Definition 6.4: Power

Power is the rate at which work is done:

$$P_{\text{avg}} = \frac{W}{\Delta t}, \qquad P = \frac{\mathrm{d}W}{\mathrm{d}t}. \tag{6.9}$$

The SI unit of power is the watt (W = J/s).

Derivation of $P = \boldsymbol{F} \cdot \boldsymbol{v}$

This important formula connects power directly to force and velocity without needing to compute the work first. Starting from $\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \boldsymbol{F} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{F} \cdot \boldsymbol{v}.$$

In one dimension, $P = Fv$. This result is valid for *any* force, constant or not: it gives the instantaneous power delivered by the force $\boldsymbol{F}$ to an object moving with velocity $\boldsymbol{v}$ at that instant.

Units and Conversions

Key Point 6.4: Common Power Units

While the SI unit of power is the watt, several other units appear frequently:

- 1 horsepower (hp) = 746 W (originally defined by James Watt to compare steam engines to draft horses).
- The kilowatt-hour (kW h) is a unit of *energy*, not power: $1\,\text{kW h} = 3.6 \times 10^6\,\text{J} = 3.6\,\text{MJ}$.
- In CGS units: $1\,\text{erg/s} = 10^{-7}\,\text{W}$.

Applications of Power

The relationship $P = Fv$ is particularly useful for problems involving engines:

Key Point 6.5: Constant Power

If an engine delivers constant power P, the driving force is $F = P/v$, which decreases as speed increases. Terminal speed occurs when $F = F_{\text{resist}}$, giving $v_{\max} = P/F_{\text{resist}}$.
For velocity-dependent drag $F_d = bv^2$, the terminal speed satisfies $P = bv_T^3$, giving $v_T = (P/b)^{1/3}$.

6.3.1 Power as a Problem-Solving Tool

The relationship $P = Fv$ provides an alternative approach to kinematics when forces depend on velocity. If the engine delivers constant power P against a velocity-dependent resistive force $F_{\text{resist}}(v)$, Newton's second law reads

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{P}{v} - F_{\text{resist}}(v). \tag{6.10}$$

This is a first-order ODE in $v(t)$, which can often be separated. Note that the constant-acceleration kinematic equations *do not apply* here, because the net force depends on velocity.

Finding distance instead of time. Using $\mathrm{d}t = \mathrm{d}x/v$, we can write $m\,v\,\mathrm{d}v = [P/v - F_{\text{resist}}(v)]\,\mathrm{d}x$, which relates speed to position instead of time.

Relating total work to time. For an engine at constant power P, the total work done over time t is $W = Pt$. Combined with the work-energy theorem: $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = Pt - W_{\text{resist}}$. If the resistive force is constant (F_r), then $W_{\text{resist}} = F_r x$, providing a relationship among v, t, and x.

Example 6.3 (Constant power, no drag). A car of mass m starts from rest with constant engine power P and no resistive forces. Then $m\,\mathrm{d}v/\mathrm{d}t = P/v$, so $mv\,\mathrm{d}v = P\,\mathrm{d}t$. Integrating: $\frac{1}{2}mv^2 = Pt$, giving $v = \sqrt{2Pt/m}$. Integrating again: $x = \int_0^t v\,\mathrm{d}t' = \int_0^t \sqrt{2Pt'/m}\,\mathrm{d}t' = \frac{2}{3}\sqrt{2P/m}\,t^{3/2}$. Note the non-linear time dependence; this is fundamentally different from constant-acceleration kinematics.

Example 6.4 (Car climbing a hill). A car of mass m climbs a slope of angle θ at constant speed v. Road friction and air drag combine into a total resistive force F_r. Find the power the engine must deliver.

Solution. At constant speed, $a = 0$, so the net force along the slope vanishes. The engine force must balance both the gravitational component down the slope and the resistance:

$$F_{\text{engine}} = mg\sin\theta + F_r.$$

Since $P = Fv$ at constant velocity:

$$P = (mg\sin\theta + F_r)\,v.$$

The first term, $mgv\sin\theta$, is the power needed to gain gravitational potential energy; the second, $F_r v$, is the power dissipated by resistance. This separation is physically illuminating: even a frictionless hill would require power $P = mgv\sin\theta$, and all of that power goes into increasing the car's height (and hence its potential energy) rather than its speed.

6.3.2 Efficiency

Real machines convert input energy into useful output energy, but some energy is inevitably "lost" to friction, heat, sound, or other non-useful forms. The **efficiency** η measures the fraction of input energy (or power) that appears as useful output:

$$\eta = \frac{W_{\text{out}}}{W_{\text{in}}} = \frac{P_{\text{out}}}{P_{\text{in}}}, \qquad 0 \le \eta \le 1. \tag{6.11}$$

Efficiency is dimensionless and is often expressed as a percentage.

Example 6.5 (Motor efficiency). An electric motor draws 500 W from the power supply and lifts a 20 kg load at a steady 2.0 m/s. What is its efficiency?

Solution. Useful output power: $P_{\text{out}} = mgv = 20(9.8)(2.0) = 392\,\text{W}$. Efficiency: $\eta = 392/500 = 0.784 = 78.4\%$. The remaining 108 W is dissipated as heat in the motor windings and friction in the bearings.

Chain efficiency. When energy passes through several stages—for example, a power plant (efficiency η_1), transmission lines (η_2), and an electric motor (η_3)—the overall efficiency is the *product* of the individual efficiencies:

$$\eta_{\text{total}} = \eta_1 \cdot \eta_2 \cdot \eta_3.$$

Each stage multiplies the surviving fraction of energy by its own efficiency, so the total efficiency is always less than the smallest individual efficiency. A chain of three 90% efficient stages has overall efficiency $0.9^3 = 0.729 = 72.9\%$, not 90%.

Problems

Problem 6.1 $\star$

A 10 kg box is pushed 5.0 m across a frictionless floor by a horizontal force of 40 N. Find the work done and the final speed from rest.

Problem 6.2 $\star\star$

A force $F(x) = 6x^2 - 2x$ (in newtons, x in meters) acts on a 3.0 kg particle. Calculate the work done from $x = 1\,\text{m}$ to $x = 4\,\text{m}$.

Problem 6.3 $\star\star\star$

A block of mass m on a rough incline (angle θ, μ_k) is pushed a distance d up the incline by force $\boldsymbol{F}$ parallel to the surface. (a) Find the work done by each force. (b) Using $W_{\text{net}} = \Delta K$, find the speed after distance d from rest. (c) Under what condition on F does the block actually accelerate up the incline?

Problem 6.4 $\star\star\star\star$

A particle of mass m on a spring (k) with kinetic friction μ_k is displaced A from equilibrium and released. (a) Show the block stops permanently when $|x| \leq \mu_k mg/k$. (b) Derive the total distance traveled. (c) Compute for $m = 0.50\,\text{kg}$, $k = 200\,\text{N/m}$, $A = 0.20\,\text{m}$, $\mu_k = 0.15$, $g = 10\,\text{m/s}^2$.

Problem 6.5 $\star\star$

A 1200 kg car has engine power $P = 60\,\text{kW}$ and resistive force 600 N. (a) Maximum speed on a flat road? (b) Acceleration at 40 m/s at maximum power?

Problem 6.6 $\star\star\star$

A car of mass m with constant engine power P accelerates from rest against drag $F_d = bv^2$. (a) Show $v_T = (P/b)^{1/3}$. (b) Show the equation of motion is $m\,\mathrm{d}v/\mathrm{d}t = P/v - bv^2$. (c) Argue that the time to reach v_T is formally infinite.

Problem 6.7 $\star\star$

A force $\boldsymbol{F} = (3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}})$ N acts on a particle while it undergoes displacement $\boldsymbol{d} = (2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}})$ m. Find the work done and the angle between $\boldsymbol{F}$ and $\boldsymbol{d}$.

Problem 6.8 $\star\star\star$

A 1500 kg elevator is lifted 40 m in 60 s at constant speed. (a) What power is required? (b) If the elevator accelerates from rest at $1.0\,\text{m/s}^2$ for the first 5 s, what peak power is needed?

Problem 6.9 $\star\star\star$

A variable force $F(x) = F_0 e^{-x/d}$ acts on a particle from $x = 0$ to $x = 3d$. Find the total work done and compare to $F_0 \cdot 3d$.

Problem 6.10 $\star\star\star\star$

A motor lifts a bucket of water (total mass M) from a well of depth h using a uniform rope of mass m and length h. (a) Find the total work done against gravity. (b) If the motor delivers constant power P, find the time required. (This requires a differential equation since the mass being lifted changes.)

Chapter 7

Kinetic Energy and the Work-Energy Theorem

In the previous chapter we introduced **work**—the mechanism by which a force transfers energy to or from an object (and **power**) the rate at which that transfer occurs. But we have not yet answered the most fundamental question: *what happens to the energy that is transferred?* Where does it go?

The answer, in the simplest case, is that it becomes **kinetic energy**—the energy of motion. In this chapter we formalize the relationship between the net work done on an object and its change in kinetic energy: the **work-energy theorem**. This theorem is not a new law of physics; it is a direct mathematical consequence of Newton's second law, repackaged in the language of energy and work. Its power lies in the fact that it replaces a vector equation ($\boldsymbol{F} = m\boldsymbol{a}$) with a scalar equation ($W_{\text{net}} = \Delta K$), eliminating the need to track directions and often bypassing the need for kinematics entirely.

Together with potential energy (Chapter 8), the work-energy theorem will provide us with a complete **energy method** for solving mechanics problems, an approach that is often far more efficient than direct application of Newton's laws, and one that generalizes to virtually every branch of physics.

7.1 Kinetic Energy

Definition and Basic Properties

Definition 7.1: Kinetic Energy

The kinetic energy of an object of mass m moving with speed v is

$$\boxed{K = \tfrac{1}{2}mv^2.} \tag{7.1}$$

Kinetic energy is always non-negative and has SI units of joules ($\text{J} = \text{kg}\,\text{m}^2/\text{s}^2$). It is a *scalar*—it depends on speed, not on the direction of motion.

Several features of this definition deserve emphasis:

- **Non-negativity:** Since $m > 0$ and $v^2 \geq 0$, we always have $K \geq 0$. Kinetic energy is zero if and only if the object is at rest ($v = 0$). There is no such thing as "negative kinetic energy."
- **Scalar nature:** Unlike momentum $\boldsymbol{p} = m\boldsymbol{v}$, which is a vector and depends on the direction of motion, kinetic energy depends only on the *magnitude* of the velocity. Two objects of the same mass moving at the same speed in opposite directions have the same kinetic energy but opposite momenta.
- **Quadratic speed dependence:** Because $K \propto v^2$, doubling the speed *quadruples* the kinetic energy; tripling the speed increases K by a factor of nine. This nonlinear scaling has profound practical consequences, as we discuss below.

The Quadratic Dependence and Its Consequences

The fact that kinetic energy scales as v^2, not v, is one of the most consequential results in mechanics. Consider the following implications:

Stopping distance. A car traveling at speed v_0 that brakes with constant friction force f will stop after a distance $d = mv_0^2/(2f)$ (as we will derive shortly using the work-energy theorem). Because $d \propto v_0^2$, doubling your highway speed from 60 km/h to 120 km/h does not double the stopping distance: it *quadruples* it. At the higher speed, you need four times the distance to stop, and a collision releases four times the energy.

Energy cost of speed. To accelerate a car from 0 m/s to 10 m/s requires an energy investment of $\frac{1}{2}m(10)^2 = 50m$ joules. To accelerate from 10 m/s to 20 m/s requires $\frac{1}{2}m(20)^2 - \frac{1}{2}m(10)^2 = 150m$ joules—three times as much energy for the same Δv. Each additional increment of speed costs more energy than the last.

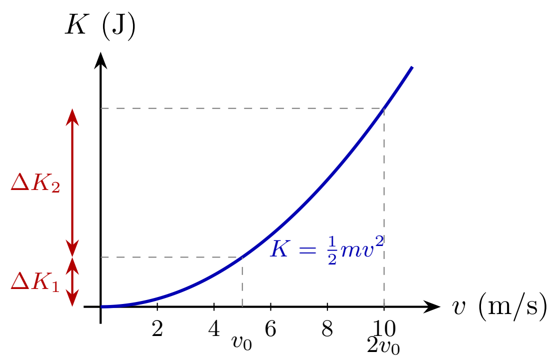


Figure 7.1.1: Kinetic energy vs. speed for a particle of mass m. Equal increments in speed ($0 \to v_0$ and $v_0 \to 2v_0$) correspond to very unequal increments in kinetic energy: $\Delta K_2 = 3\,\Delta K_1$. This is a direct consequence of the v^2 dependence.

Kinetic Energy in Component Form

When the velocity is given in Cartesian components, $\boldsymbol{v} = v_x\,\hat{\boldsymbol{\imath}} + v_y\,\hat{\boldsymbol{\jmath}} + v_z\,\hat{\boldsymbol{k}}$, the speed is $v = |\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$, and the kinetic energy becomes

$$K = \tfrac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \tfrac{1}{2}m\,\boldsymbol{v}\cdot\boldsymbol{v}. \tag{7.2}$$

This is simply half the mass times the dot product of the velocity with itself. Note that each component contributes independently: $K = K_x + K_y + K_z$ where $K_x = \frac{1}{2}mv_x^2$, etc. This decomposition is sometimes useful in projectile motion problems, where the horizontal and vertical components of velocity evolve independently.

Kinetic Energy Is Frame-Dependent

Because velocity depends on the observer's reference frame, so does kinetic energy. A ball sitting on the tray table of a moving airplane has zero kinetic energy in the airplane's frame but substantial kinetic energy in the ground frame. This is not a flaw: the work-energy theorem $W_{\text{net}} = \Delta K$ holds in every inertial frame, and both sides of the equation transform in exactly the same way. The physics is self-consistent in any inertial frame.

Key Point 7.1: Frame Dependence of Kinetic Energy

The numerical value of an object's kinetic energy depends on the reference frame. However, the *change* in kinetic energy produced by a given process is physically meaningful, and the work-energy theorem correctly relates it to the net work in every inertial frame.

Kinetic Energy of a System of Particles

For a system of N particles, the total kinetic energy is simply the sum:

$$K_{\text{total}} = \sum_{i=1}^{N} \tfrac{1}{2} m_i v_i^2. \tag{7.3}$$

A powerful decomposition due to König states that this total kinetic energy can be split into two parts:

$$K_{\text{total}} = \underbrace{\tfrac{1}{2} M v_{\text{cm}}^2}_{\text{CM translation}} + \underbrace{K_{\text{int}}}_{\text{internal motion}}, \tag{7.4}$$

where $M = \sum m_i$ is the total mass, v_{cm} is the speed of the center of mass, and K_{int} is the kinetic energy of all particles measured in the center-of-mass frame. We will prove this result (König's theorem) in Chapter 10 when we discuss the center-of-mass reference frame; for now, we simply note that it exists and that it will play an important role in our treatment of collisions.

Historical Note: *Vis Viva*

The quantity mv^2 (without the factor of $\frac{1}{2}$) was introduced by Gottfried Wilhelm Leibniz in 1686 under the name *vis viva* (Latin for "living force"). Leibniz argued that mv^2, not mv, was the correct measure of the "quantity of motion," in opposition to the Cartesian view. The factor of $\frac{1}{2}$ was introduced later by Gaspard-Gustave de Coriolis in 1829, who defined work as Fd and showed that the theorem relating work to kinetic energy takes its simplest form with $K = \frac{1}{2}mv^2$. The modern terminology "kinetic energy" (from the Greek *kinesis*, meaning "motion") was coined by Lord Kelvin in 1849.

Worked Examples

Example 7.1 (Comparing kinetic energies). A truck of mass $M = 3000\,\text{kg}$ moves at $v_1 = 20\,\text{m/s}$. A car of mass $m = 1000\,\text{kg}$ moves at $v_2 = 40\,\text{m/s}$. (a) Which has greater kinetic energy? (b) Which requires more work to stop?

Solution. (a) Truck: $K_{\text{truck}} = \frac{1}{2}(3000)(20)^2 = 6.0 \times 10^5\,\text{J} = 600\,\text{kJ}$. Car: $K_{\text{car}} = \frac{1}{2}(1000)(40)^2 = 8.0 \times 10^5\,\text{J} = 800\,\text{kJ}$. Despite having three times the mass, the truck has *less* kinetic energy because its speed is only half that of the car, and $K \propto v^2$.

(b) By the work-energy theorem ($W_{\text{net}} = \Delta K$), the work required to bring an object from speed v to rest is $W = 0 - K = -K$. The magnitude of the required work equals the kinetic energy. The car requires more work to stop: $|W_{\text{car}}| = 800\,\text{kJ} > |W_{\text{truck}}| = 600\,\text{kJ}$.

Example 7.2 (Kinetic energy at different points of a projectile trajectory). A ball of mass m is thrown at speed v_0 at angle θ above the horizontal. Neglecting air resistance, find the kinetic energy at the highest point.

Solution. At the highest point, the vertical component of velocity is zero ($v_y = 0$) and only the horizontal component remains: $v_x = v_0 \cos\theta$. Therefore,

$$K_{\text{top}} = \tfrac{1}{2}mv_x^2 = \tfrac{1}{2}mv_0^2\cos^2\theta = K_0\cos^2\theta,$$

where $K_0 = \tfrac{1}{2}mv_0^2$ is the initial kinetic energy. For a launch angle of 45°, the kinetic energy at the top is exactly half the initial kinetic energy. The "missing" kinetic energy has been converted into gravitational potential energy, a connection we will formalize in Chapter 8.

Key Point 7.2: Kinetic Energy Is Never Zero in Projectile Motion (for $\theta < 90°$)

A projectile launched at angle $\theta < 90°$ always retains some kinetic energy because the horizontal component of velocity is never zero (in the absence of air resistance). Only a ball launched straight up ($\theta = 90°$) has $K = 0$ at the top.

7.2 The Work-Energy Theorem

The work-energy theorem is the central result of this chapter. It connects the net work done on an object (a quantity computed from the forces) to the change in the object's kinetic energy, a quantity computed from the motion.

Theorem 7.1: Work-Energy Theorem

The net work done on an object equals the change in its kinetic energy:

$$\boxed{W_{\text{net}} = \Delta K = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2.} \tag{7.5}$$

If $W_{\text{net}} > 0$, the object speeds up. If $W_{\text{net}} < 0$, the object slows down. If $W_{\text{net}} = 0$, the speed is unchanged (though the direction may change).

Derivation from Newton's Second Law (1D, Constant Force)

The simplest derivation uses the kinematic equation $v_f^2 = v_i^2 + 2a\Delta x$, which holds for constant acceleration in one dimension. Multiplying both sides by $\tfrac{1}{2}m$:

$$\tfrac{1}{2}mv_f^2 = \tfrac{1}{2}mv_i^2 + ma\,\Delta x.$$

Since $F_{\text{net}} = ma$ (constant), the term $ma\,\Delta x = F_{\text{net}}\Delta x = W_{\text{net}}$, giving

$$\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = W_{\text{net}}.$$

This derivation is instructive because it shows that the work-energy theorem is not an independent principle but a direct consequence of Newton's second law combined with kinematics.

Derivation (1D, Variable Force)

The constant-force derivation above is limited. For a force that varies with position, we need calculus. Starting from Newton's second law, $F_{\text{net}} = ma = m\dfrac{\mathrm{d}v}{\mathrm{d}t}$, the net work done from x_i to x_f is:

$$W_{\text{net}} = \int_{x_i}^{x_f} F_{\text{net}}\,\mathrm{d}x = \int_{x_i}^{x_f} m\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}x. \tag{7.6}$$

The key step is to change the variable of integration from position to velocity. Since $\mathrm{d}x = v\,\mathrm{d}t$, we have $\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}x = v\,\mathrm{d}v$, and the integral becomes:

$$W_{\text{net}} = \int_{v_i}^{v_f} mv\,\mathrm{d}v = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \Delta K. \tag{7.7}$$

This derivation is completely general for one-dimensional motion: it makes no assumption about the form of $F_{\text{net}}(x)$. The force can be an arbitrary function of position, time, or velocity, and the result still holds.

Key Point 7.3: The Change-of-Variable Trick

The substitution $\mathrm{d}x = v\,\mathrm{d}t$, which converts $\int F\,\mathrm{d}x$ into $\int mv\,\mathrm{d}v$, is one of the most important mathematical maneuvers in introductory mechanics. It will appear repeatedly: in deriving the work done by springs, in power problems, and in the relativistic generalization. The essential insight is that it eliminates time from the problem entirely, relating force directly to speed.

Derivation (3D, General)

The one-dimensional derivation generalizes straightforwardly to three dimensions. Starting from $\boldsymbol{F}_{\text{net}} = m\boldsymbol{a} = m\dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$, the net work along a path C is:

$$W_{\text{net}} = \int_C \boldsymbol{F}_{\text{net}} \cdot \mathrm{d}\boldsymbol{r} = \int_{t_i}^{t_f} m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{v}\,\mathrm{d}t.$$

We used $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$. Now we invoke the product-rule identity:

$$\boldsymbol{a} \cdot \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{v} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v} \cdot \boldsymbol{v}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(v^2). \tag{7.8}$$

This identity follows from the product rule for dot products: $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v} \cdot \boldsymbol{v}) = \dot{\boldsymbol{v}} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \dot{\boldsymbol{v}} = 2\,\boldsymbol{a} \cdot \boldsymbol{v}$. Substituting:

$$W_{\text{net}} = \int_{t_i}^{t_f} \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}(v^2)\,\mathrm{d}t = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \Delta K.$$

This is the work-energy theorem in full generality. It holds for *any* path, *any* force law (constant, variable, velocity-dependent, time-dependent), and in *any* number of dimensions. The only requirement is that Newton's second law holds—i.e., that we are in an inertial reference frame.

Common Mistake 7.1: The WET Requires Net Work

The work-energy theorem uses the *net* work done on the object: the total work done by *all* forces. Do not confuse W_{net} with the work done by any single force. Omitting a force (such as friction or the normal force on a moving surface) will give an incorrect result.

What the Work-Energy Theorem Does (and Does Not) Tell You

The work-energy theorem is a scalar equation. This is both its strength and its limitation:

Strengths:

- It eliminates the need to track vector directions: only the *speed* appears, not the velocity vector.
- It eliminates time as a variable: the theorem relates force and displacement directly to the change in speed, without requiring knowledge of $v(t)$ or $a(t)$.
- It applies to *any* force law, including forces that vary with position, velocity, or time.

Limitations:

- It tells you the *speed* at the final point, but not the *direction* of motion. For that, you need Newton's second law or conservation of momentum.
- It gives no information about the *time* required for the process. A car that accelerates from 0 to 60 mph does the same amount of work regardless of whether it takes 4 s or 10 s; the work-energy theorem cannot distinguish these cases (for that, you need power).
- It applies to a *single particle* or, more precisely, to the motion of a single point. Extending it to deformable bodies requires additional care (see the discussion of internal energy in Chapter 8).

Graphical Interpretation

In one dimension, the net work $W_{\text{net}} = \int_{x_i}^{x_f} F_{\text{net}}(x)\,dx$ is the **signed area** under the $F_{\text{net}}(x)$ curve. By the work-energy theorem, this area equals the change in kinetic energy ΔK.

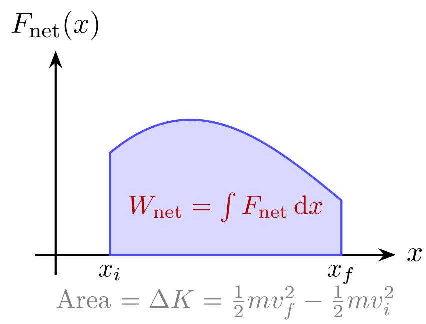


Figure 7.2.1: The net work done on an object equals the area under the $F_{\text{net}}(x)$ curve, which by the work-energy theorem equals ΔK.

This graphical interpretation is particularly powerful when $F_{\text{net}}(x)$ is given as a graph rather than a formula (see Problem 7.10).

Worked Examples

Example 7.3 (Braking distance—the v^2 law). A car of mass m traveling at speed v_0 on a flat road brakes with a constant friction force f. Find the stopping distance.

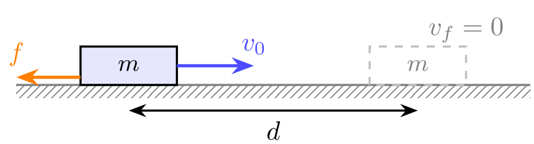


Figure 7.2.2: A car braking to a stop. The friction force f opposes the displacement.

Solution. The only horizontal force is friction, which opposes the motion: $W_{\text{net}} = -fd$. The car starts at $v_i = v_0$ and ends at $v_f = 0$. Applying the work-energy theorem:

$$W_{\text{net}} = \Delta K \implies -fd = 0 - \tfrac{1}{2}mv_0^2 \implies \boxed{d = \frac{mv_0^2}{2f}}.$$

Since $d \propto v_0^2$, doubling the speed quadruples the stopping distance. For a 1500 kg car braking with $f = \mu_k mg = 0.7(1500)(9.8) = 10\,290\,\text{N}$:

At $v_0 = 30\,\text{m/s}$ ($\approx 108\,\text{km/h}$): $d = \frac{1500(30)^2}{2(10290)} = 65.5\,\text{m}$.

At $v_0 = 60\,\text{m/s}$ ($\approx 216\,\text{km/h}$): $d = \frac{1500(60)^2}{2(10290)} = 262\,\text{m}$—four times as far.

This is why speed limits exist, and why the consequences of speeding grow so rapidly.

Example 7.4 (Block on a frictionless incline). A block of mass $m = 2.0\,\text{kg}$ starts from rest and slides $d = 3.0\,\text{m}$ down a frictionless incline of angle $\theta = 30°$. Find the speed at the bottom using the work-energy theorem (not kinematics).

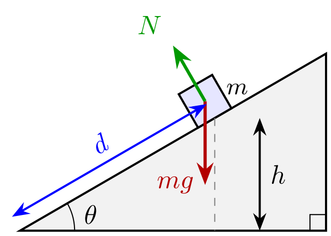


Figure 7.2.3: A block sliding down a frictionless incline of angle θ.

Solution. We compute the work done by each force:

Gravity: The displacement is d along the incline. The component of gravity along the incline is $mg\sin\theta$. So $W_g = mgd\sin\theta = mgh$, where $h = d\sin\theta$ is the vertical drop.

Normal force: Perpendicular to the displacement, so $W_N = 0$.

The net work is $W_{\text{net}} = mgh$. Applying the work-energy theorem with $v_i = 0$:

$$mgh = \tfrac{1}{2}mv_f^2 - 0 \implies v_f = \sqrt{2gh} = \sqrt{2gd\sin\theta}.$$

Substituting: $v_f = \sqrt{2(9.8)(3.0)\sin 30°} = \sqrt{29.4} = 5.42\,\text{m/s}$.

Note that this result depends only on the *height* $h = d\sin\theta$, not on the mass or the angle separately. This is a hint that something deeper is going on: the work done by gravity depends only on the change in height, a property that will lead us to the concept of gravitational potential energy in Chapter 8.

Example 7.5 (Variable force: polynomial). A force $F(x) = 3x^2 + 2$ (in newtons, with x in meters) acts on a 1.0 kg particle that starts from rest at $x = 0$. Find the speed at $x = 2.0\,\text{m}$.

Solution. Since this is the only force (and hence the net force), the net work is:

$$W_{\text{net}} = \int_0^2 (3x^2 + 2)\,\mathrm{d}x = \left[x^3 + 2x\right]_0^2 = (8 + 4) - 0 = 12\,\text{J}.$$

By the work-energy theorem with $v_i = 0$:

$$12\,\text{J} = \tfrac{1}{2}(1.0)\,v_f^2 \implies v_f = \sqrt{24} = 4.90\,\text{m/s}.$$

Example 7.6 (Spring launch). A spring with spring constant $k = 500\,\text{N/m}$, compressed by $x_0 = 0.10\,\text{m}$, launches a ball of mass $m = 0.50\,\text{kg}$ horizontally on a frictionless surface. Find the launch speed.

Solution. The work done by the spring as it relaxes from $x_i = -x_0$ to $x_f = 0$ (its natural length) is:

$$W_{\text{spring}} = \int_{-x_0}^{0} (-kx)\,\mathrm{d}x = -\frac{1}{2}k\big[x^2\big]_{-x_0}^{0} = \frac{1}{2}kx_0^2.$$

(The spring force $F = -kx$ is negative when $x < 0$, meaning it pushes the ball in the $+x$ direction: the force does positive work.)

With no other horizontal forces and $v_i = 0$:

$$\tfrac{1}{2}kx_0^2 = \tfrac{1}{2}mv_f^2 \quad\Longrightarrow\quad v_f = x_0\sqrt{k/m} = 0.10\sqrt{500/0.50} = 0.10 \times 31.6 = 3.16\,\text{m/s}.$$

Key Point 7.4: Springs Store Energy

A compressed (or stretched) spring stores energy $U_s = \frac{1}{2}kx^2$. When released, it converts this stored energy into kinetic energy. This is the first example of a broader principle: **potential energy**, which we develop systematically in Chapter 8.

Example 7.7 (Connected system—Atwood-like machine). Two blocks ($m_1 = 3\,\text{kg}$ on a table with $\mu_k = 0.25$; $m_2 = 5\,\text{kg}$ hanging) are connected by a light string over a massless, frictionless pulley. Using the work-energy theorem, find the speed after m_2 falls $d = 2.0\,\text{m}$ from rest.

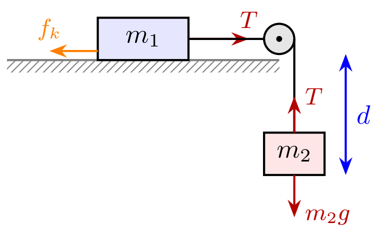


Figure 7.2.4: Two blocks connected by a string over a pulley. When m_2 descends a distance d, block m_1 moves the same distance d across the table.

Solution. The system consists of both blocks. Since the string is inextensible, both blocks move with the same speed v after m_2 has descended d. We apply the work-energy theorem to the entire system.

The work done by each force:

- *Gravity on m_2:* $W_{g,2} = m_2gd = 5(9.8)(2) = 98\,\text{J}$ (positive; gravity acts downward on m_2, which moves downward).
- *Gravity on m_1:* $W_{g,1} = 0$ (m_1 moves horizontally).
- *Normal force on m_1:* $W_N = 0$ (perpendicular to motion).
- *Friction on m_1:* $W_f = -\mu_k m_1 gd = -0.25(3)(9.8)(2) = -14.7\,\text{J}$.
- *Tension:* The tension does positive work Td on m_1 and negative work $-Td$ on m_2. These cancel: $W_T = Td - Td = 0$. *Internal forces in a system connected by an inextensible string do zero net work.*

The net work on the system is $W_{\text{net}} = 98 - 14.7 = 83.3\,\text{J}$. The total kinetic energy is:

$$\Delta K = \tfrac{1}{2}(m_1 + m_2)v^2 - 0 = \tfrac{1}{2}(8)v^2.$$

Setting $W_{\text{net}} = \Delta K$:

$$83.3 = 4v^2 \implies v = \sqrt{83.3/4} = 4.56\,\text{m/s}.$$

Key Point 7.5: Tension in Inextensible Strings Does Zero Net Work on the System

When two objects are connected by a light, inextensible string, the tension does positive work on one object and equal negative work on the other. The net work by tension on the *system* is zero. This is why we can analyze the system as a whole without needing to find the tension.

Example 7.8 (Finding an unknown force). A 0.145 kg baseball traveling at 40 m/s is caught by a fielder whose glove recoils 0.15 m during the catch. Estimate the average force exerted on the ball.

Solution. The ball decelerates from $v_i = 40\,\text{m/s}$ to $v_f = 0$ over a distance $d = 0.15\,\text{m}$. By the work-energy theorem:

$$W_{\text{net}} = \Delta K \implies -\bar{F}\,d = 0 - \tfrac{1}{2}mv_i^2$$

$$\bar{F} = \frac{mv_i^2}{2d} = \frac{0.145(40)^2}{2(0.15)} = \frac{232}{0.30} = 773\,\text{N}.$$

This is over 500 times the ball's weight, a large but brief force. The larger the recoil distance d, the smaller the average force. This is why catchers pull their glove back during a catch: by increasing d, they reduce $\bar{F}$, making the catch less painful.

Example 7.9 (Loop-the-loop: minimum height). A small block of mass m starts from rest at height h on a frictionless track and enters a circular loop of radius R. Find the minimum height h from which the block can complete the loop.

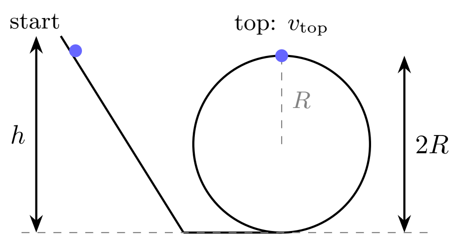


Figure 7.2.5: A block released from height h enters a circular loop of radius R.

Solution. At the top of the loop, the block is at height $2R$. The work-energy theorem (with only gravity doing work, since the track is frictionless and the normal force is always perpendicular to the motion) gives:

$$W_{\text{net}} = mg(h - 2R) = \tfrac{1}{2}mv_{\text{top}}^2 - 0 \implies v_{\text{top}}^2 = 2g(h - 2R).$$

At the top of the loop, the block must maintain contact with the track. Both gravity (mg) and the normal force (N) point toward the center (downward), providing the centripetal acceleration:

$$mg + N = \frac{mv_{\text{top}}^2}{R}.$$

The minimum speed occurs when $N = 0$ (the track barely pushes on the block), giving $v_{\text{top,min}}^2 = gR$. Substituting into the energy equation:

$$gR = 2g(h_{\min} - 2R) \implies h_{\min} = 2R + \frac{R}{2} = \frac{5R}{2}.$$

The block must start at least $\frac{5}{2}R$ above the bottom of the loop to complete it. Note that this result is independent of the mass m and of the shape of the ramp leading into the loop, all that matters is the starting height.

Deriving Kinematic Equations from the Work-Energy Theorem

It is instructive to see that the work-energy theorem is *equivalent* to the time-free kinematic equation for constant acceleration. For a constant net force F, the acceleration is $a = F/m$ and the net work is $W = Fd$. The WET gives:

$$Fd = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 \quad \Longrightarrow \quad \frac{F}{m}(2d) = v_f^2 - v_i^2 \quad \Longrightarrow \quad v_f^2 = v_i^2 + 2ad.$$

This is exactly the kinematic equation $v^2 = v_0^2 + 2a\Delta x$. The work-energy theorem and the kinematics of constant acceleration are two faces of the same coin, but the work-energy theorem is more general because it applies even when the acceleration is not constant.

7.3 Applications of the Work-Energy Theorem

The work-energy theorem is most powerful in situations where tracking individual forces and accelerations via Newton's second law would be tedious or impractical. Below we develop several important application categories.

Problems with Multiple Forces

When several forces act simultaneously, the work-energy theorem offers two equivalent routes to the final speed:

Method 1: Sum individual works. Compute $W_1, W_2, \ldots$ for each force, then set $\sum W_i = \Delta K$.

Method 2: Use the net force. Find $\boldsymbol{F}_{\text{net}}$ first, then compute $W_{\text{net}} = \int \boldsymbol{F}_{\text{net}} \cdot \mathrm{d}\boldsymbol{r}$.

Method 1 is usually preferred because it keeps track of each force's individual contribution, which is essential for the energy methods developed in Chapter 8.

Example 7.10 (Block pushed up a rough incline). A block of mass m is pushed a distance d up a rough incline (angle θ, kinetic friction coefficient μ_k) by a force F parallel to the surface. The block starts from rest. Find the final speed.

Solution. The forces acting on the block along the incline are: the applied force F (up the incline, positive work), gravity component $mg\sin\theta$ (down the incline, negative work), and friction $f_k = \mu_k mg\cos\theta$ (down the incline, negative work). The normal force is perpendicular to the displacement and does zero work.

Work by each force:

$$W_F = Fd, \qquad W_g = -mgd\sin\theta, \qquad W_f = -\mu_k mgd\cos\theta.$$

Net work: $W_{\text{net}} = Fd - mgd\sin\theta - \mu_k mgd\cos\theta$. The work-energy theorem gives:

$$\tfrac{1}{2}mv^2 = Fd - mgd(\sin\theta + \mu_k\cos\theta)$$

$$\boxed{v = \sqrt{\frac{2d}{m}\big[F - mg(\sin\theta + \mu_k\cos\theta)\big]}.}$$

The block actually accelerates only if the radicand is positive, i.e., $F > mg(\sin\theta + \mu_k \cos\theta)$. If F is exactly equal to this threshold, all of the applied work goes into fighting gravity and friction, and the block moves at constant speed ($v = 0$ throughout if starting from rest, but this would mean it cannot actually move, so more precisely the block moves at constant speed if given an initial push and F equals the threshold value).

Velocity-Dependent Forces

Forces that depend on velocity (such as fluid drag) present a challenge for the standard kinematic approach because the acceleration is not constant. The work-energy theorem, combined with the substitution $F\,\mathrm{d}x = mv\,\mathrm{d}v$, handles these problems elegantly.

Example 7.11 (Linear drag). A particle of mass m subject to a linear drag force $F(v) = -bv$ is projected at speed v_0. Find the total distance to stop.

Solution. The net force is $F = -bv$, which depends on velocity. Using $F\,\mathrm{d}x = mv\,\mathrm{d}v$:

$$-bv\,\mathrm{d}x = mv\,\mathrm{d}v \quad\Longrightarrow\quad -b\,\mathrm{d}x = m\,\mathrm{d}v \quad \text{(dividing by } v\text{, valid for } v > 0\text{).}$$

Integrating from $v = v_0$ to $v = 0$:

$$-b\int_0^d \mathrm{d}x = m\int_{v_0}^0 \mathrm{d}v \quad\Longrightarrow\quad -bd = -mv_0 \quad\Longrightarrow\quad \boxed{d = \frac{mv_0}{b}.}$$

Note that the stopping distance is proportional to v_0, *not* v_0^2. This is fundamentally different from the constant-friction case ($d \propto v_0^2$). The reason is that linear drag weakens as the particle slows, so the deceleration decreases over time, and the particle takes longer (in both time and distance) to reach each successive lower speed.

For comparison, the velocity as a function of time is $v(t) = v_0 e^{-bt/m}$, which only asymptotically approaches zero. The total distance is $d = \int_0^\infty v_0 e^{-bt/m}\,\mathrm{d}t = mv_0/b$, confirming our result.

Example 7.12 (Quadratic drag). A particle of mass m subject to quadratic drag $F(v) = -cv^2$ starts at speed v_0. Find the speed after traveling a distance d.

Solution. Using $F\,\mathrm{d}x = mv\,\mathrm{d}v$:

$$-cv^2\,\mathrm{d}x = mv\,\mathrm{d}v \quad\Longrightarrow\quad -cv\,\mathrm{d}x = m\,\mathrm{d}v.$$

But this mixes v and x. Instead, go back to $\int F\,\mathrm{d}x = \Delta K$ more directly. We need to express everything in terms of one variable. Using $v\,\mathrm{d}v = a\,\mathrm{d}x = (F/m)\,\mathrm{d}x = (-cv^2/m)\,\mathrm{d}x$:

$$\frac{\mathrm{d}v}{v} = -\frac{c}{m}\,\mathrm{d}x.$$

Integrating: $\ln(v_f/v_0) = -cd/m$, giving

$$\boxed{v_f = v_0\, e^{-cd/m}.}$$

The speed decreases exponentially with distance traveled. Unlike the linear drag case, the particle never fully stops (in finite distance or finite time), though practically it becomes negligibly slow.

Work-Energy Theorem for Systems

When applying the work-energy theorem to a system of connected objects (pulleys, Atwood machines, blocks on inclines connected by ropes), we have two choices:

Strategy 7.1: System Approach vs. Individual Approach

System approach: Apply $W_{\text{net,ext}} = \Delta K_{\text{total}}$ to the whole system. Internal forces (tensions in inextensible strings) cancel and need not be computed.
Individual approach: Apply $W_{\text{net}} = \Delta K$ to each object separately. Tensions appear but can be found by solving the resulting equations simultaneously.
Use the system approach when you need only the speed; use the individual approach when you also need the tension or other internal forces.

Example 7.13 (Atwood machine via the WET). Two masses m_1 and $m_2 > m_1$ hang from a massless string over a frictionless pulley. Find the speed after m_2 has descended a distance h from rest.

Solution (system approach). Both masses move with the same speed v. When m_2 descends h, mass m_1 ascends h. The only external force doing work is gravity:

$$W_{\text{net}} = m_2 gh - m_1 gh = (m_2 - m_1)gh.$$

The total kinetic energy is $\Delta K = \frac{1}{2}(m_1 + m_2)v^2$. Therefore:

$$(m_2 - m_1)gh = \tfrac{1}{2}(m_1 + m_2)v^2 \quad \Longrightarrow \quad \boxed{v = \sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2}}.}$$

This can also be derived from Newton's second law: the acceleration is $a = (m_2 - m_1)g/(m_1 + m_2)$ and then $v^2 = 2ah$, which gives the same result—confirming the equivalence of the two approaches.

Determining Unknown Forces from Energy Considerations

The work-energy theorem can be "run in reverse": if you know the change in kinetic energy and the distance, you can infer the average force. This is especially useful for impulsive processes (collisions, catches, impacts) where the force varies rapidly and its detailed time-dependence is unknown.

Example 7.14 (Bullet penetration). A bullet of mass $m = 10\,\text{g}$ traveling at $v_0 = 400\,\text{m/s}$ embeds itself $d = 0.12\,\text{m}$ into a wooden block. Find the average resistive force.

Solution. The bullet decelerates from v_0 to 0. By the work-energy theorem:

$$-\bar{F}\,d = 0 - \tfrac{1}{2}mv_0^2 \quad \Longrightarrow \quad \bar{F} = \frac{mv_0^2}{2d} = \frac{0.010 \times (400)^2}{2(0.12)} = \frac{1600}{0.24} \approx 6700\,\text{N}.$$

This is a large force, but it acts over a very short distance. The work-energy theorem gives us this average force without needing to know how the resistive force varies with depth.

Key Point 7.6: Average Force from the WET

For any process that brings an object from speed v to rest over distance d, the average resistive force is $\bar{F} = mv^2/(2d)$. This applies to braking, bullet penetration, crumple zones in cars, landing on cushions—any situation where energy is absorbed over a finite distance. *Increasing d always decreases $\bar{F}$*, which is the physical principle behind safety engineering: padding, airbags, and crumple zones all work by increasing the stopping distance.

7.4 The Three Forms of the Work-Energy Theorem

The work-energy theorem as stated in Theorem 7.1 accounts for *all* forces through the net work W_{net}. In practice, it is often useful to classify forces as **conservative** or **non-conservative** and treat them differently. This classification leads to three equivalent formulations of the energy principle, each suited to different types of problems.

We will develop the concept of conservative forces and potential energy in full detail in Chapter 8; for now, we state the key definitions and results needed to see the three forms side by side.

Conservative and Non-Conservative Forces (Preview)

A force is **conservative** if the work it does depends only on the initial and final positions, not on the path. Equivalently, the work around any closed loop is zero. Gravity and the spring force are conservative; kinetic friction and air drag are not.

For any conservative force, we can define a **potential energy** U such that $W_{\text{cons}} = -\Delta U$. This converts the work done by the conservative force into a change in a state function, a function that depends only on position.

The Three Forms

Strategy 7.2: Energy Method Selection

1. **Work-Energy Theorem** ($W_{\text{net}} = \Delta K$): Always valid. Every force's work appears on the left side. No potential energy is used. Best when all forces are known and none have conveniently defined potential energies, or when you want to track each force's contribution individually.

2. **Conservation of Mechanical Energy** ($K_i + U_i = K_f + U_f$): Valid *only* when no non-conservative forces do work ($W_{\text{nc}} = 0$). This is the simplest form when applicable: no forces need to be computed explicitly, only initial and final energies. Cannot be used when friction, drag, or applied forces are present.

3. **Generalized Work-Energy Theorem** ($W_{\text{nc}} = \Delta K + \Delta U = \Delta E_{\text{mech}}$): Always valid. Conservative forces are handled implicitly through ΔU; only the work of non-conservative forces (W_{nc}) appears on the left. This is the most practical form for problems involving both conservative and non-conservative forces.

The logical relationship among these forms is:

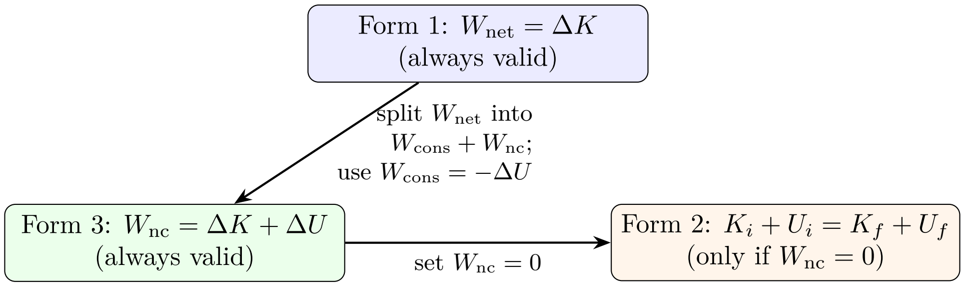


Figure 7.4.1: The logical hierarchy of the three energy formulations. Form 1 is the most general; Form 3 separates conservative forces into potential energy; Form 2 is the special case with no non-conservative work.

Deriving Form 3 from Form 1. Split the net work into contributions from conservative and non-conservative forces: $W_{\text{net}} = W_{\text{cons}} + W_{\text{nc}}$. Since $W_{\text{cons}} = -\Delta U$:

$$-\Delta U + W_{\text{nc}} = \Delta K \quad \Longrightarrow \quad W_{\text{nc}} = \Delta K + \Delta U = \Delta E_{\text{mech}}.$$

Deriving Form 2 from Form 3. If $W_{\text{nc}} = 0$ (no friction, no applied forces, no drag):

$$0 = \Delta K + \Delta U \quad \Longrightarrow \quad K_i + U_i = K_f + U_f.$$

We will use Forms 2 and 3 extensively in Chapter 8, once we have developed the full machinery of potential energy.

Example 7.15 (Choosing the right form). A block slides down a frictionless ramp from height h. Using each of the three forms, find the speed at the bottom.

Form 1: $W_{\text{net}} = W_g + W_N = mgh + 0 = \frac{1}{2}mv^2$, so $v = \sqrt{2gh}$.

Form 2: $K_i + U_i = K_f + U_f \Rightarrow 0 + mgh = \frac{1}{2}mv^2 + 0$, so $v = \sqrt{2gh}$. ✓

Form 3: $W_{\text{nc}} = 0$ (no non-conservative forces), so $\Delta K + \Delta U = 0 \Rightarrow \frac{1}{2}mv^2 - mgh = 0$, so $v = \sqrt{2gh}$. ✓

All three forms give the same answer, as they must. For this particular problem, Form 2 is the simplest.

7.5 Limitations, Subtleties, and Deeper Connections

The WET Applies to Particles, Not General Bodies

The work-energy theorem as derived in this chapter applies strictly to point particles or rigid bodies in translation. For deformable bodies—a ball that compresses upon impact, a spring that stretches, a car whose crumple zone collapses—part of the work done goes into changing the *internal energy* of the body (deformation, heat, sound) rather than changing the kinetic energy of the center of mass.

The correct generalization, which we will encounter in later chapters, involves tracking the work done on the *center of mass* separately from the work that changes the internal state:

$$W_{\text{net,external}} = \Delta K_{\text{cm}} + \Delta E_{\text{internal}}.$$

For a rigid body, $\Delta E_{\text{internal}} = 0$, and we recover the simple WET. For a deformable body, some of the work goes into heating or deforming the object, and the center-of-mass kinetic energy changes by less than the total work done.

Frame Dependence Revisited

We noted earlier that kinetic energy is frame-dependent. Let us see explicitly that the work-energy theorem remains valid in different frames.

Consider a ball of mass m dropped from rest in the ground frame. After falling a height h, the ball has speed $v = \sqrt{2gh}$ and kinetic energy $K = mgh$. The net work done (by gravity) is $W = mgh = \Delta K$. ✓

Now consider the same process from a frame moving at speed u to the right. In this frame, the ball initially has velocity $(u, 0)$, and after falling height h it has velocity $(u, -\sqrt{2gh})$. The kinetic energies are:

$$K_i = \tfrac{1}{2}mu^2, \qquad K_f = \tfrac{1}{2}m(u^2 + 2gh) = \tfrac{1}{2}mu^2 + mgh.$$

So $\Delta K = mgh$—exactly the same as in the ground frame. The work done by gravity is also the same: $W_g = mgh$ (gravity acts downward, displacement is h downward). The theorem holds in both frames.

This is not a coincidence. The work done by a force depends on the displacement, and in this case the vertical displacement is the same in both frames (only the horizontal motion differs, and gravity does no horizontal work). More generally, one can show that W_{net} and ΔK transform identically under Galilean boosts, so the theorem holds in every inertial frame.

Relativistic Kinetic Energy: A Preview

The formula $K = \frac{1}{2}mv^2$ is an approximation valid only when $v \ll c$, where $c \approx 3 \times 10^8$ m/s is the speed of light. The correct relativistic expression, derived from Einstein's special theory of relativity, is:

$$K_{\text{rel}} = (\gamma - 1)mc^2, \qquad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{7.9}$$

At low speeds, the binomial approximation gives $\gamma \approx 1 + \frac{1}{2}v^2/c^2$, so:

$$K_{\text{rel}} \approx \left(\frac{1}{2}\frac{v^2}{c^2}\right)mc^2 = \frac{1}{2}mv^2,$$

recovering the classical result. The leading relativistic correction is:

$$K_{\text{rel}} \approx \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \cdots$$

At $v/c = 0.1$ (about 3×10^7 m/s), the correction is roughly 0.75%—small but measurable in precision experiments. At speeds close to c, the relativistic kinetic energy diverges to infinity, implying that no massive object can reach the speed of light. This is entirely outside the scope of classical mechanics, but it is worth knowing where the classical formula breaks down and why.

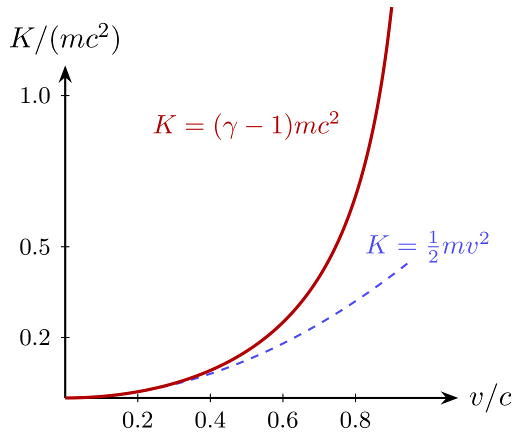


Figure 7.5.1: Classical (dashed blue) and relativistic (solid red) kinetic energy as functions of v/c. The two agree at low speeds, but the relativistic curve diverges as $v \to c$, imposing an absolute speed limit.

Connection to Lagrangian Mechanics

In more advanced formulations of mechanics, kinetic energy plays a central structural role. In the **Lagrangian formulation** (developed by Joseph-Louis Lagrange in 1788), the dynamics of a system are encoded in the **Lagrangian** $\mathcal{L} = K - U$, and the equations of motion follow from requiring that the *action* $S = \int \mathcal{L}\, \mathrm{d}t$ be stationary (Hamilton's principle). The work-energy theorem emerges naturally from this variational framework. We mention this to foreshadow the remarkable, and ultimately fundamental, role that kinetic and potential energy play well beyond the force-based approach of this course.

Problems

Problem 7.1 ⋆

A 5.0 kg object moves at 4.0 m/s. (a) Find K. (b) What net work is required to bring it to rest? (c) What net work is required to double its speed? (d) What net work is required to increase its speed from 4.0 m/s to 6.0 m/s? Compare with part (c) and comment.

Problem 7.2 ⋆

A 60 kg runner and a 1200 kg car are both moving at 10 m/s. (a) Compare their kinetic energies. (b) The runner accelerates to 12 m/s and the car to 12 m/s. Compare the *changes* in kinetic energy. Why is the change for the car so much larger, even though both experienced the same Δv?

Problem 7.3 ⋆⋆

A 2.0 kg block slides 3.0 m from rest down a frictionless 30° incline. Find the speed at the bottom using the work-energy theorem (not kinematics). Verify by computing v from $v^2 = v_0^2 + 2a\Delta x$ and showing the results agree.

Problem 7.4 ⋆⋆

A force $F(x) = 3x^2 + 2$ N acts on a 1.0 kg particle starting from rest at $x = 0$. (a) Find the speed at $x = 2.0$ m. (b) At what position does the particle first reach 3.0 m/s?

Problem 7.5 ⋆⋆

A baseball pitcher throws a 0.145 kg ball, accelerating it from rest over a distance of approximately 1.5 m (the length of the pitching motion). If the ball leaves the hand at 40 m/s, estimate the average force exerted by the pitcher on the ball.

Problem 7.6 ⋆⋆

A 0.25 kg ball is dropped from a height of 2.0 m and bounces back to a height of 1.5 m. (a) Find the kinetic energy just before impact. (b) Find the kinetic energy just after the bounce. (c) How much energy was lost during the collision with the floor? (d) What fraction of the kinetic energy was lost?

Problem 7.7 ⋆ ⋆ ⋆

A spring ($k = 500$ N/m) compressed 0.10 m launches a 0.50 kg ball vertically. (a) Find the launch speed. (b) Find the maximum height. (c) If instead the spring launches the ball along a 45° frictionless ramp, find the distance along the ramp the ball travels before stopping.

Problem 7.8 ⋆ ⋆ ⋆

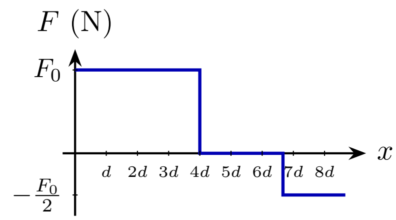
Two blocks ($m_1 = 3$ kg on a table, $\mu_k = 0.25$; $m_2 = 5$ kg hanging) are connected by a string over a massless, frictionless pulley. (a) Using the work-energy theorem applied to the system, find the speed after m_2 falls 2.0 m from rest. (b) Now find the tension in the string. (*Hint:* apply the WET to m_2 alone.)

Problem 7.9 ⋆ ⋆ ⋆

Show that the work-energy theorem is consistent with $v^2 = v_0^2 + 2a\Delta x$ for a constant net force. That is, starting from $W_{\text{net}} = \Delta K$ with $W_{\text{net}} = F\Delta x$ and $F = ma$, derive the kinematic equation.

Problem 7.10 ⋆⋆⋆

The force shown in the graph below acts on a 2.0 kg particle that starts from rest at $x = 0$. Find the speed at $x = 4d$, $x = 6d$, and $x = 8d$.



Problem 7.11 ⋆⋆⋆

A 50 kg skier starts from rest at the top of a 20 m high frictionless hill, slides down, and enters a circular loop of radius $R = 6.0\,\text{m}$. (a) Find the speed at the bottom of the loop. (b) Find the speed at the top of the loop. (c) Find the normal force on the skier at the top of the loop. (d) What is the minimum hill height for the skier to complete the loop?

Problem 7.12 ⋆⋆⋆⋆

A block on a rough surface (μ_k) is projected at speed v_0. (a) Derive the stopping distance using the WET. (b) By what factor does it increase if the initial speed doubles? (c) Derive the average power dissipated by friction during the entire stopping process.

Problem 7.13 ⋆⋆⋆⋆

A particle of mass m subject to the force $F(v) = -bv$ starts at speed v_0. (a) Using $F\,\mathrm{d}x = mv\,\mathrm{d}v$, find the total distance to stop. (b) Using $F = m\,\mathrm{d}v/\mathrm{d}t$, find $v(t)$ and show that $v(t) = v_0 e^{-bt/m}$. (c) Show that $d = \int_0^\infty v(t)\,\mathrm{d}t$ gives the same total distance as part (a).

Problem 7.14 ⋆⋆⋆⋆

A particle of mass m moves under a position-dependent force $F(x) = F_0 e^{-x/\lambda}$, starting from rest at $x = 0$. (a) Find the work done from $x = 0$ to $x = L$. (b) Find the speed at $x = L$. (c) Find the terminal speed v_∞ as $L \to \infty$ and interpret physically. (d) At what distance has the particle acquired 90% of its terminal kinetic energy?

Problem 7.15 ⋆⋆⋆⋆

A particle of mass m moves under $F(x) = F_0 \sin(\pi x/L)$ from $x = 0$ to $x = L$, starting from rest. (a) Find the work done. (b) Find the speed at $x = L$. (c) At what position is the speed a maximum? (*Hint:* the speed is maximized where $\mathrm{d}K/\mathrm{d}x = 0$, which is where $F(x) = 0$ with F changing from positive to negative. But be careful—F is positive on the entire interval $[0, L]$.)

Problem 7.16 ⋆⋆⋆⋆⋆

Prove the three-dimensional work-energy theorem. Starting from $\boldsymbol{F}_{\text{net}} = m\boldsymbol{a}$, show that $\int_C \boldsymbol{F}_{\text{net}} \cdot \mathrm{d}\boldsymbol{r} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$, where the integral is along the particle's actual trajectory. (*Hint:* use $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$ and the identity $\boldsymbol{a} \cdot \boldsymbol{v} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(v^2)$.)

Problem 7.17 ⋆⋆⋆⋆⋆

(*Relativistic work-energy theorem.*) In special relativity, Newton's second law becomes $F = \frac{\mathrm{d}p}{\mathrm{d}t}$ with $p = \gamma mv$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. (a) Show that $F\,\mathrm{d}x = c^2\,\mathrm{d}(\gamma m)$ by using $F\,\mathrm{d}x = v\,\mathrm{d}p$.

(b) Integrate to obtain $W = (\gamma_f - \gamma_i)mc^2$, recovering the relativistic kinetic energy $K = (\gamma - 1)mc^2$ when the particle starts from rest. (c) Taylor-expand for $v \ll c$ to recover $K \approx \frac{1}{2}mv^2$.

Chapter 8

Potential Energy and Energy Conservation

In Chapter 7 we established that the net work done on an object equals its change in kinetic energy. That result is always valid, but it has a practical limitation: every force must be accounted for explicitly in the work calculation, even forces like gravity that always give the same result regardless of the path.

In this chapter we exploit that path-independence. For a special class of forces called **conservative forces**, we can define a **potential energy** function $U(\boldsymbol{r})$ that encodes all the work information in a single scalar field. This transforms the work-energy theorem into the principle of **conservation of mechanical energy**: in the absence of friction and other non-conservative forces, the total mechanical energy $E = K + U$ is constant throughout the motion.

This is one of the most powerful problem-solving tools in all of physics. Instead of tracking forces, accelerations, and kinematics step by step, we simply compare the energy at two points. The approach generalizes far beyond mechanics—conservation of energy, in one form or another, underlies every branch of physics.

8.1 Conservative and Non-Conservative Forces

Path Independence: The Defining Property

Definition 8.1: Conservative Force

A force $\boldsymbol{F}$ is **conservative** if the work it does on an object moving between two points A and B depends only on A and B, not on the path taken:

$$W_{A\to B} = \int_A^B \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} \quad \text{is the same for } \textit{every} \text{ path from } A \text{ to } B. \tag{8.1}$$

Equivalently, the work done by $\boldsymbol{F}$ around any closed loop is zero:

$$\oint \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = 0. \tag{8.2}$$

These two conditions—path independence (8.1) and the closed-loop test (8.2)—are mathematically equivalent. To see why, suppose the work from A to B is path-independent. Consider any closed path: travel from A to B along path 1, then back from B to A along path 2. The total work around the loop is $W_1 + W_2$. But $W_2 = -W_{2,\text{reversed}}$, where $W_{2,\text{reversed}}$ is the work from A to B along path 2. Since $W_1 = W_{2,\text{reversed}}$ (path independence), the loop integral vanishes. The converse argument is identical.

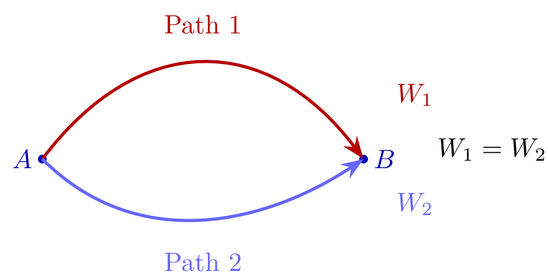


Figure 8.1.1: A conservative force does the same work along any path from A to B. Equivalently, the work around the closed loop (Path 1 forward, Path 2 backward) is zero.

Examples of Conservative Forces

Gravity (near Earth's surface). The gravitational force $\boldsymbol{F}_g = -mg\,\hat{\boldsymbol{j}}$ does work $W_g = -mg\,\Delta y = -mg(y_f - y_i)$ that depends only on the change in height, regardless of any horizontal motion or the shape of the path. This is what we observed in Example 6.1: a block descending a height h by any route (straight down, along a ramp, over a hill) always has $W_g = mgh$.

The spring force. For an ideal spring obeying Hooke's law, $F = -kx$, the work done in moving from x_i to x_f is $W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$, which depends only on the endpoints x_i and x_f.

Universal gravitation. The gravitational force $\boldsymbol{F} = -\frac{Gm_1m_2}{r^2}\,\hat{\boldsymbol{r}}$ between two point masses is conservative. Its work depends only on the initial and final separations r_i and r_f.

Electrostatic force. Coulomb's law, $\boldsymbol{F} = \frac{kq_1q_2}{r^2}\,\hat{\boldsymbol{r}}$, has the same mathematical structure as universal gravitation and is also conservative.

Non-Conservative Forces

A force is **non-conservative** if its work *does* depend on the path. The paradigmatic example is kinetic friction.

Why friction is non-conservative. Consider sliding a box from A to B along a straight path of length d, then back from B to A along the same path. Friction does negative work on each leg: $W_{\text{friction}} = -f_k d$ going and $-f_k d$ returning, for a total of $-2f_k d \neq 0$. The work around this closed loop is negative, violating the closed-loop condition (8.2).

More generally, kinetic friction always opposes the direction of motion, so it always does negative work. A round trip always dissipates energy regardless of the path, and longer paths dissipate more energy. This path-length dependence is the hallmark of a non-conservative force.

Key Point 8.1: Conservative vs. Non-Conservative: The Key Distinction

For a **conservative force**, work depends only on endpoints: we can store it as potential energy.

For a **non-conservative force**, work depends on the path: we cannot define a potential energy, and energy is typically converted to heat or other "non-mechanical" forms.

The mathematical criterion (proved in Appendix B.3) is: a force $\boldsymbol{F}(\boldsymbol{r})$ that depends only on position is conservative if and only if $\nabla \times \boldsymbol{F} = \boldsymbol{0}$. In one dimension, *every* position-dependent force $F(x)$ is automatically conservative.

Common Mistake 8.1: Velocity-Dependent Forces

A force that depends on velocity (like air drag $\boldsymbol{F} = -b\boldsymbol{v}$) is generally non-conservative. Such forces cannot even be tested by the curl criterion $\nabla \times \boldsymbol{F} = \boldsymbol{0}$, which applies only to forces that depend solely on position. Velocity-dependent forces extract or inject energy in a path-dependent way.

Verifying Path Independence: A Worked Example

Example 8.1 (Two paths under gravity). A 2 kg ball moves from point A at coordinates $(0, 5\,\mathrm{m})$ to point B at $(3, 0)$ by two different routes: (i) straight down to $(0, 0)$ then horizontally to $(3, 0)$; (ii) diagonally from A to B. Show that gravity does the same work in both cases.

Solution. The gravitational force is $\boldsymbol{F}_g = -mg\,\hat{\boldsymbol{\jmath}}$.

Path (i): Along the vertical segment, $\mathrm{d}\boldsymbol{r} = \mathrm{d}y\,\hat{\boldsymbol{\jmath}}$, so $W_1 = \int_5^0 (-mg)\,\mathrm{d}y = mg(5) = 98\,\mathrm{J}$. Along the horizontal segment, $\mathrm{d}\boldsymbol{r} = \mathrm{d}x\,\hat{\boldsymbol{\imath}}$, so $W_2 = 0$ (force perpendicular to displacement). Total: $W = 98\,\mathrm{J}$.

Path (ii): Parameterize the diagonal by $\boldsymbol{r}(t) = (3t,\ 5(1-t))$ for $t \in [0, 1]$. Then $\mathrm{d}\boldsymbol{r} = (3\,\hat{\boldsymbol{\imath}} - 5\,\hat{\boldsymbol{\jmath}})\,\mathrm{d}t$ and $W = \int_0^1 (-mg\,\hat{\boldsymbol{\jmath}}) \cdot (3\,\hat{\boldsymbol{\imath}} - 5\,\hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^1 5mg\,\mathrm{d}t = 5mg = 98\,\mathrm{J}$.

Both paths give $W = mgh = mg \times 5 = 98\,\mathrm{J}$. Gravity is conservative: only the height change matters.

Example 8.2 (A non-conservative force). The force $\boldsymbol{F} = y\,\hat{\boldsymbol{\imath}}$ acts on a particle moving from the origin $(0, 0)$ to the point $(1, 1)$. Compute the work along (i) the path $y = x$ and (ii) the path $y = x^2$. Are they equal?

Solution. Along any path, $W = \int F_x\,\mathrm{d}x + F_y\,\mathrm{d}y = \int y\,\mathrm{d}x + 0$ (since $F_y = 0$).

Path (i): $y = x$, so $W = \int_0^1 x\,\mathrm{d}x = \frac{1}{2}$.

Path (ii): $y = x^2$, so $W = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$.

The two results differ ($\frac{1}{2} \neq \frac{1}{3}$), so this force is non-conservative. One can verify: $\partial F_x/\partial y = 1$ but $\partial F_y/\partial x = 0$, so $(\nabla \times \boldsymbol{F})_z = \partial F_y/\partial x - \partial F_x/\partial y = -1 \neq 0$.

8.2 Potential Energy

Definition

The path-independence of conservative forces has a remarkable consequence: we can encode the work done by a conservative force entirely in a scalar function of position called **potential energy**.

Definition 8.2: Potential Energy

For a conservative force $\boldsymbol{F}$, the **potential energy** $U(\boldsymbol{r})$ is defined by the relation

$$\boxed{W_{\mathrm{cons}} = -\Delta U = -(U_f - U_i) = U_i - U_f.} \tag{8.3}$$

Equivalently, the potential energy difference between two points is:

$$U(\boldsymbol{r}_B) - U(\boldsymbol{r}_A) = -\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}. \tag{8.4}$$

The sign convention is crucial: positive work by the conservative force *decreases* potential energy, and vice versa.

The definition specifies only *differences* in potential energy, not the absolute value. We are free to choose the reference point where $U = 0$—what matters physically is always ΔU.

Key Point 8.2: Only Differences in Potential Energy Matter

The choice of where $U = 0$ is a convention, like choosing the origin of a coordinate system. Different choices lead to different values of U at each point, but the *difference* $\Delta U = U_f - U_i$ is always the same. No physical prediction ever depends on the absolute value of potential energy, only on changes.

Deriving the Common Potential Energies

We now derive the three most important potential energy functions in mechanics from their defining forces.

1. Gravitational potential energy (near Earth's surface).

The gravitational force near Earth's surface is $\boldsymbol{F}_g = -mg\,\hat{\boldsymbol{\jmath}}$. The work done from height y_i to height y_f along any path is:

$$W_g = \int \boldsymbol{F}_g \cdot \mathrm{d}\boldsymbol{r} = \int (-mg)\,\mathrm{d}y = -mg(y_f - y_i) = -mg\,\Delta y.$$

Since $W_g = -\Delta U_g$, we identify:

$$\boxed{U_g = mgy.} \tag{8.5}$$

The reference point $U_g = 0$ is at $y = 0$, which we choose for convenience (often the ground or the lowest point in the problem).

2. Elastic (spring) potential energy.

The spring force is $F = -kx$, where x is the displacement from the natural length. The work done by the spring from x_i to x_f is:

$$W_s = \int_{x_i}^{x_f} (-kx)\,\mathrm{d}x = -\frac{1}{2}k(x_f^2 - x_i^2).$$

Setting $W_s = -\Delta U_s$, we identify:

$$\boxed{U_s = \tfrac{1}{2}kx^2.} \tag{8.6}$$

The reference point $U_s = 0$ is at $x = 0$ (the natural length of the spring). Note that $U_s \geq 0$ always: the spring stores energy whether compressed ($x < 0$) or stretched ($x > 0$).

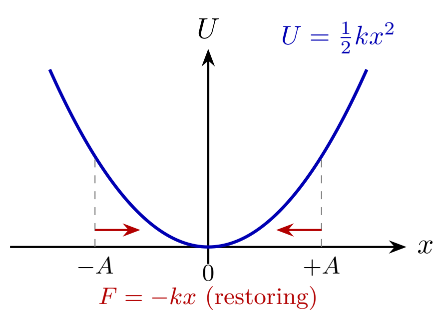


Figure 8.2.1: The elastic potential energy $U = \frac{1}{2}kx^2$. The force (red arrows) always pushes toward $x = 0$, the minimum of U.

3. Universal gravitational potential energy.

The gravitational force between masses m_1 and m_2 separated by distance r is $\boldsymbol{F} = -\frac{Gm_1m_2}{r^2}\,\hat{\boldsymbol{r}}$. To find $U(r)$, we compute the work done as the separation changes from r_i to r_f along a radial path:

$$W = \int_{r_i}^{r_f} \left(-\frac{Gm_1m_2}{r^2}\right) \mathrm{d}r = Gm_1m_2\left(\frac{1}{r_f} - \frac{1}{r_i}\right).$$

Setting $W = -\Delta U = -(U_f - U_i)$:

$$U_f - U_i = -Gm_1m_2\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = Gm_1m_2\left(\frac{1}{r_i} - \frac{1}{r_f}\right).$$

Choosing $U \to 0$ as $r \to \infty$ (the conventional reference point), we set $r_i \to \infty$ so $U_i = 0$:

$$\boxed{U_G = -\frac{Gm_1m_2}{r}.} \tag{8.7}$$

The minus sign is physically significant: gravity is attractive, so bringing two masses closer (r decreasing) lowers the potential energy. Two masses infinitely far apart have $U = 0$; bringing them together releases energy.

Theorem 8.1: Summary of Common Potential Energies

Gravitational (near surface, reference at $y = 0$): $U_g = mgy$.
Elastic (spring, reference at natural length): $U_s = \frac{1}{2}kx^2$.
Gravitational (universal, reference at $r \to \infty$): $U_G = -\dfrac{Gm_1m_2}{r}$.

Force from Potential Energy: The Gradient Relationship

The definition $W = -\Delta U$ can be inverted: if we know $U(\boldsymbol{r})$, we can recover the force. In one dimension, the work done over an infinitesimal displacement is $\mathrm{d}W = F\,\mathrm{d}x = -\mathrm{d}U$, giving:

$$\boxed{F(x) = -\frac{\mathrm{d}U}{\mathrm{d}x}.} \tag{8.8}$$

The force is the *negative slope* of the potential energy curve. Where U rises steeply, the force is large and directed "downhill"; where U is flat, the force vanishes.

In three dimensions, the generalization involves the gradient:

$$\boxed{\boldsymbol{F} = -\nabla U = -\frac{\partial U}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial U}{\partial y}\hat{\boldsymbol{\jmath}} - \frac{\partial U}{\partial z}\hat{\boldsymbol{k}}.} \tag{8.9}$$

The force points in the direction of *steepest decrease* of U, and its magnitude equals the rate of that decrease.

Example 8.3 (Recovering forces from potentials).

(a) From $U_g = mgy$: $F_y = -\mathrm{d}(mgy)/\mathrm{d}y = -mg$, recovering the familiar downward gravitational force.

(b) From $U_s = \frac{1}{2}kx^2$: $F = -\mathrm{d}(\frac{1}{2}kx^2)/\mathrm{d}x = -kx$, recovering Hooke's law.

(c) From $U_G = -Gm_1m_2/r$: since in spherical coordinates $F_r = -\mathrm{d}U/\mathrm{d}r = -Gm_1m_2/r^2$ (i.e., attractive and inverse-square), recovering Newton's law of gravitation.

(d) From $U(x, y) = \alpha(x^2 + 4y^2)$: $F_x = -2\alpha x$, $F_y = -8\alpha y$. The force points toward the origin and is stronger in the y-direction. Verify: $\nabla \times \boldsymbol{F} = 0$, confirming this force is conservative.

Key Point 8.3: The Potential Energy–Force Duality

Given a conservative force, we can integrate to find U. Given U, we can differentiate to find the force. The two descriptions are completely equivalent, related by

$$\boldsymbol{F} = -\nabla U \qquad \Longleftrightarrow \qquad U(\boldsymbol{r}) = -\int_{\text{ref}}^{r} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}'.$$

The potential energy description is often more compact and more powerful, a single scalar function encodes the full vector force field.

8.3 Conservation of Mechanical Energy

Derivation from the Work-Energy Theorem

We now combine the work-energy theorem with the concept of potential energy to obtain one of the most important results in physics.

Start with the work-energy theorem: $W_{\text{net}} = \Delta K$. Split the net work into contributions from conservative and non-conservative forces:

$$W_{\text{net}} = W_{\text{cons}} + W_{\text{nc}}.$$

Since $W_{\text{cons}} = -\Delta U$ for each conservative force:

$$-\Delta U + W_{\text{nc}} = \Delta K \qquad \Longrightarrow \qquad W_{\text{nc}} = \Delta K + \Delta U.$$

Now define the **total mechanical energy**:

$$E_{\text{mech}} = K + U. \tag{8.10}$$

Then:

$$\boxed{W_{\text{nc}} = \Delta E_{\text{mech}} = (K_f + U_f) - (K_i + U_i).} \tag{8.11}$$

This is the **generalized work-energy theorem**: non-conservative forces change the total mechanical energy.

In the special case where no non-conservative forces do work ($W_{\text{nc}} = 0$):

Theorem 8.2: Conservation of Mechanical Energy

If only conservative forces do work on a system, the total mechanical energy $E = K + U$ is conserved:

$$\boxed{K_i + U_i = K_f + U_f.} \tag{8.12}$$

Equivalently, $\Delta E = \Delta K + \Delta U = 0$: any increase in kinetic energy comes at the expense of potential energy, and vice versa.

Interpretation

Conservation of mechanical energy is a *bookkeeping principle*: energy is neither created nor destroyed but merely converted between kinetic and potential forms. A ball thrown upward converts kinetic energy into gravitational potential energy on the way up and reconverts it on the way down. A mass on a spring oscillates between kinetic and elastic potential energy. At every instant, $K + U$ remains the same constant E.

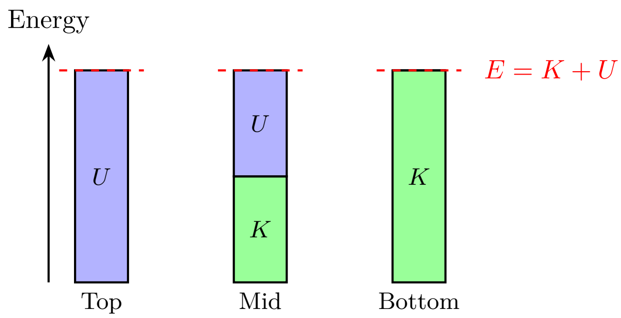


Figure 8.3.1: Energy bar charts for a ball falling from rest. The total $E = K + U$ (dashed red line) remains constant; energy converts from potential (blue) to kinetic (green).

Problem-Solving Strategy

Strategy 8.1: Using Conservation of Energy

1. **Identify the system** and all forces acting on it. Classify each force as conservative (with a defined potential energy) or non-conservative.
2. **Choose reference points** for each potential energy (where is $U = 0$?).
3. **Identify the initial and final states**—specify K and U at each.
4. **Apply the appropriate energy equation:**
   - If $W_{\text{nc}} = 0$: use $K_i + U_i = K_f + U_f$.
   - If $W_{\text{nc}} \neq 0$: use $W_{\text{nc}} = (K_f + U_f) - (K_i + U_i)$.
5. **Solve** for the unknown quantity.

The beauty of this method is that you never need to find accelerations, resolve forces into components, or integrate equations of motion. You compare two snapshots (initial and final) and the energy equation does the rest.

Worked Examples: Conservative Systems ($W_{\text{nc}} = 0$)

Example 8.4 (Free fall from energy conservation). A ball of mass m is dropped from height h. Find the speed just before it hits the ground.

Solution. Choose the ground as $y = 0$ (so $U_g = 0$ at the ground). Initial state: $K_i = 0$ (starts from rest), $U_i = mgh$. Final state: $K_f = \frac{1}{2}mv^2$, $U_f = 0$. Conservation of energy:

$$0 + mgh = \tfrac{1}{2}mv^2 + 0 \quad \Longrightarrow \quad v = \sqrt{2gh}.$$

This is the same result we obtained from kinematics ($v^2 = v_0^2 + 2gh$) and from the work-energy theorem ($W_g = \Delta K$), but the energy method is the simplest: no vectors, no kinematics, just two snapshots.

Example 8.5 (Pendulum speed at the bottom). A pendulum of length ℓ and mass m is released from rest at angle θ_0 from the vertical. Find the speed at the lowest point and the tension there.

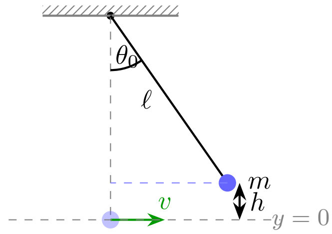


Figure 8.3.2: A pendulum released from angle θ_0. The height of the mass above the lowest point is $h = \ell(1 - \cos\theta_0)$.

Solution. The height of the bob above the lowest point is $h = \ell - \ell\cos\theta_0 = \ell(1-\cos\theta_0)$. Choose $U = 0$ at the lowest point. The tension in the string does no work (it is always perpendicular to the velocity), so $W_{\text{nc}} = 0$.

Energy conservation:

$$0 + mg\ell(1-\cos\theta_0) = \tfrac{1}{2}mv_{\text{bot}}^2 + 0 \quad\Longrightarrow\quad v_{\text{bot}} = \sqrt{2g\ell(1-\cos\theta_0)}.$$

For the tension at the bottom, we need Newton's second law in the radial direction. At the lowest point, the net upward force provides the centripetal acceleration:

$$T - mg = \frac{mv_{\text{bot}}^2}{\ell} = \frac{m\cdot 2g\ell(1-\cos\theta_0)}{\ell} = 2mg(1-\cos\theta_0).$$

$$\boxed{T = mg(3 - 2\cos\theta_0).}$$

For small angles, $\cos\theta_0 \approx 1 - \theta_0^2/2$, so $T \approx mg(1+\theta_0^2)$—only slightly above mg, as expected for a gently swinging pendulum.

Example 8.6 (Roller coaster). A roller-coaster car of mass m starts from rest at the top of a hill of height h_1 and coasts along a frictionless track over a second hill of height $h_2 < h_1$. Find the speed at the top of the second hill.

Solution. There are no non-conservative forces (frictionless, and the normal force does no work). With $U = 0$ at the base:

$$0 + mgh_1 = \tfrac{1}{2}mv^2 + mgh_2 \quad\Longrightarrow\quad v = \sqrt{2g(h_1 - h_2)}.$$

The speed depends only on the *height difference*, not on the shape of the track, the mass of the car, or the distances involved. This is the power of energy conservation.

Example 8.7 (Loop-the-loop). A block of mass m is released from rest at height h on a frictionless track and enters a vertical circular loop of radius R. Find (a) the minimum h for the block to complete the loop, and (b) the normal force at the top of the loop when $h = 3R$.

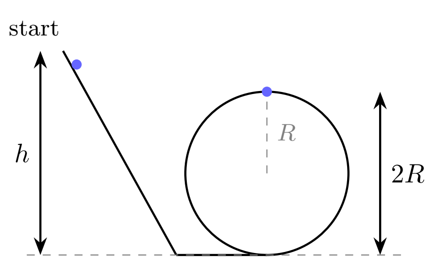


Figure 8.3.3: A block released from height h enters a circular loop of radius R.

Solution. (a) At the top of the loop, the block is at height $2R$. Energy conservation:

$$mgh = mg(2R) + \tfrac{1}{2}mv_{\text{top}}^2 \implies v_{\text{top}}^2 = 2g(h - 2R).$$

At the top of the loop, gravity and the normal force both point downward (toward the center):

$$mg + N = \frac{mv_{\text{top}}^2}{R}.$$

The minimum condition is $N = 0$ (the track barely pushes on the block): $v_{\text{top,min}}^2 = gR$. Substituting:

$$gR = 2g(h_{\text{min}} - 2R) \implies h_{\text{min}} = \frac{5R}{2}.$$

(b) When $h = 3R$: $v_{\text{top}}^2 = 2g(3R - 2R) = 2gR$. Newton's second law at the top:

$$mg + N = \frac{m(2gR)}{R} = 2mg \implies N = mg.$$

The normal force at the top equals the weight. For comparison, at $h = h_{\text{min}} = 5R/2$, we had $N = 0$.

Example 8.8 (Spring launcher). A spring with constant k, compressed by Δx, launches a block of mass m up a frictionless ramp inclined at angle θ. How far along the ramp does the block travel?

Solution. At the launch point: $K_i = 0$, $U_s = \frac{1}{2}k(\Delta x)^2$, $U_g = 0$. At the highest point (distance d up the ramp): $K_f = 0$, $U_s = 0$, $U_g = mgd\sin\theta$. Conservation:

$$\tfrac{1}{2}k(\Delta x)^2 = mgd\sin\theta \implies \boxed{d = \frac{k(\Delta x)^2}{2mg\sin\theta}.}$$

Note that d is independent of the shape of the transition from spring to ramp, only the initial and final states matter.

8.4 The Generalized Work-Energy Theorem

When non-conservative forces are present (friction, air drag, applied forces from engines) mechanical energy is not conserved. Instead, the non-conservative work W_{nc} accounts for the change in mechanical energy:

$$\boxed{W_{\text{nc}} = \Delta E_{\text{mech}} = (K_f + U_f) - (K_i + U_i).} \tag{8.13}$$

For dissipative forces like friction, $W_{\text{nc}} < 0$, and mechanical energy *decreases*. The "lost" mechanical energy is converted into thermal energy (heat), sound, deformation, etc. For applied forces (a motor, a person pushing), W_{nc} can be positive, increasing mechanical energy.

Where Does the Energy Go?

When you slide a book across a table and it comes to rest, the kinetic energy is gone. Where did it go? Friction converted it to **thermal energy**—the book and table are slightly warmer. In the fully general energy principle:

$$\Delta K + \Delta U + \Delta E_{\text{thermal}} + \Delta E_{\text{other}} = 0, \tag{8.14}$$

where ΔE_{other} includes sound, deformation, chemical energy, etc. Energy is always conserved in the universe; "non-conservation of mechanical energy" merely means that some mechanical energy was converted to non-mechanical forms.

Key Point 8.4: Conservation of Energy: The Universal Principle

Mechanical energy conservation ($K + U = \text{const}$) is a special case of a far deeper principle: **total energy is always conserved**. When mechanical energy decreases due to friction, the "missing" energy appears as thermal energy. No process in the universe has ever been observed to violate conservation of total energy. This principle, the **first law of thermodynamics**, is one of the most fundamental laws of nature.

Worked Examples with Non-Conservative Forces

Example 8.9 (Sliding down a rough ramp). A block of mass m starts from rest at height h on a ramp inclined at angle θ with kinetic friction coefficient μ_k. Find the speed at the bottom.

Solution. The distance along the ramp is $d = h/\sin\theta$. The normal force is $N = mg\cos\theta$, so friction does work $W_f = -\mu_k N d = -\mu_k mg\cos\theta \cdot h/\sin\theta = -\mu_k mgh\cot\theta$.

Generalized WET:

$$W_{\text{nc}} = (K_f + U_f) - (K_i + U_i).$$

$$-\mu_k mgh\cot\theta = (\tfrac{1}{2}mv^2 + 0) - (0 + mgh).$$

$$v = \sqrt{2gh(1 - \mu_k\cot\theta)}.$$

The block reaches the bottom only if $\mu_k\cot\theta < 1$, i.e., $\mu_k < \tan\theta$. If $\mu_k \geq \tan\theta$, friction is strong enough to prevent the block from sliding at all (as we know from statics: $\mu_s \geq \tan\theta$ is the condition for equilibrium on an incline).

Example 8.10 (Ramp to rough floor). A block of mass m slides from rest down a frictionless ramp of height h onto a rough horizontal surface with coefficient μ_k. How far does the block slide before stopping?

Solution. Initial state: at the top of the ramp, $K_i = 0$, $U_i = mgh$. Final state: at rest on the floor, $K_f = 0$, $U_f = 0$. The only non-conservative work is from friction on the horizontal surface: $W_f = -\mu_k mg \cdot d$.

Generalized WET:

$$-\mu_k mgd = (0 + 0) - (0 + mgh) \quad\Longrightarrow\quad d = \frac{h}{\mu_k}.$$

The stopping distance is proportional to the starting height and inversely proportional to the friction coefficient. Note that the mass cancels, as it does in all problems involving gravity and friction together.

Example 8.11 (Child on a slide with friction). A child of mass $m = 25\,\text{kg}$ slides down a $3.0\,\text{m}$ long slide inclined at $40°$. She starts from rest and reaches the bottom at $3.2\,\text{m/s}$. Find the friction force.

Solution. The height is $h = d\sin\theta = 3.0\sin 40° = 1.93\,\text{m}$. The generalized WET gives:

$$W_f = (K_f + U_f) - (K_i + U_i) = (\tfrac{1}{2}mv^2 + 0) - (0 + mgh) = \tfrac{1}{2}(25)(3.2)^2 - 25(9.8)(1.93).$$

$$W_f = 128 - 472.9 = -344.9\,\text{J}.$$

Since $W_f = -f \cdot d$: $f = 344.9/3.0 = 115\,\text{N}$.

Check: the normal force is $N = mg\cos 40° = 25(9.8)(0.766) = 187.7\,\text{N}$, giving $\mu_k = f/N = 115/187.7 = 0.61$. This is a reasonable value for a child's clothing on a metal slide.

Example 8.12 (Spring with friction). A block of mass $m = 0.50\,\text{kg}$ on a horizontal surface ($\mu_k = 0.20$) is pushed against a spring ($k = 500\,\text{N/m}$) that is compressed by $x_0 = 0.10\,\text{m}$. After release, how far from the spring's natural length does the block travel before stopping?

Solution. Take the spring's natural length as the origin. The block starts at $x = -x_0$ with $K = 0$ and ends at some position $x = d > 0$ with $K = 0$. The spring potential energy decreases from $\frac{1}{2}kx_0^2$ to 0 (assuming $d > 0$ and the block detaches from the spring at $x = 0$). Friction acts over the total distance $x_0 + d$:

$$W_f = -\mu_k mg(x_0 + d).$$

Generalized WET with $K_i = K_f = 0$:

$$-\mu_k mg(x_0 + d) = (0 + 0) - (0 + \tfrac{1}{2}kx_0^2).$$

$$\mu_k mg(x_0 + d) = \tfrac{1}{2}kx_0^2 \quad \Longrightarrow \quad d = \frac{kx_0^2}{2\mu_k mg} - x_0.$$

Substituting: $d = \frac{500(0.01)}{2(0.20)(0.50)(9.8)} - 0.10 = \frac{5.0}{1.96} - 0.10 = 2.55 - 0.10 = 2.45\,\text{m}.$

8.5 Energy Diagrams

For one-dimensional conservative systems, the potential energy curve $U(x)$ is an extraordinarily powerful tool. It encodes the complete qualitative dynamics of the system (allowed regions, turning points, equilibria, oscillation frequencies) all readable by inspection, without solving any differential equation.

The Basic Idea

Consider a particle of mass m moving under a conservative force $F(x) = -\mathrm{d}U/\mathrm{d}x$ with total mechanical energy $E = K + U$. Since $K = \frac{1}{2}mv^2 \geq 0$, we have:

$$K = E - U(x) \geq 0 \quad \Longrightarrow \quad U(x) \leq E. \tag{8.15}$$

The particle can only exist in regions where the potential energy curve lies at or below the total energy line. Regions where $U(x) > E$ are **classically forbidden**—the particle cannot enter them, because doing so would require negative kinetic energy (negative v^2), which is physically impossible.

Reading an Energy Diagram

Strategy 8.2: Reading Energy Diagrams

Given a plot of $U(x)$ with a horizontal line at E:

1. **Allowed regions:** The particle is confined to regions where $U(x) \leq E$. Shade or mentally mark regions where $U > E$ as forbidden.
2. **Turning points:** These occur where $U(x) = E$, i.e., where the U curve intersects the E line. At a turning point, $K = 0$ and the particle momentarily stops. It then reverses direction because the force $F = -\mathrm{d}U/\mathrm{d}x$ pushes it back toward the allowed region.
3. **Kinetic energy:** At any position x, the kinetic energy is the *vertical gap* between the E line and the U curve: $K(x) = E - U(x)$. The speed is maximum where $U(x)$ is

minimum (largest gap).

4. **Force:** The force is $F = -\mathrm{d}U/\mathrm{d}x$, i.e., the negative slope of the U curve. Where U slopes upward (to the right), the force is to the left; where U slopes downward, the force is to the right. The particle is always pushed "downhill" on the potential energy curve.

5. **Equilibria:** Points where $\mathrm{d}U/\mathrm{d}x = 0$ (the curve is flat) are equilibria: the force vanishes there. There are three types:
   - **Stable equilibrium** at a local minimum of U: $U'' > 0$. A small displacement produces a restoring force.
   - **Unstable equilibrium** at a local maximum of U: $U'' < 0$. A small displacement produces a force that pushes the particle further away.
   - **Neutral equilibrium** at an inflection point where $U' = 0$ and $U'' = 0$: the force is zero and remains zero (to leading order) for small displacements.

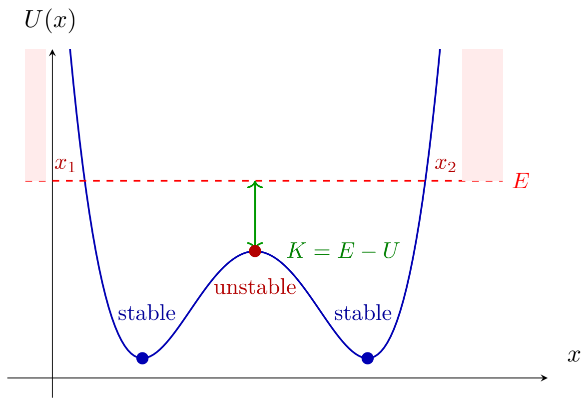


Figure 8.5.1: A potential energy diagram $U(x)$ with two stable equilibria (local minima), one unstable equilibrium (local maximum), total energy E (dashed red), and the kinetic energy $K = E - U$ shown as the vertical gap. The particle is confined between the turning points x_1 and x_2.

Bound and Unbound Motion

The total energy E determines the qualitative character of the motion:

Bound motion (E below all barriers): The particle oscillates back and forth between two turning points, trapped in a potential energy "well." At lower energies, the particle may be confined to one well; at higher energies (above the barrier), it may traverse multiple wells.

Unbound motion (E above all barriers): The particle has enough energy to escape to infinity. There may be a single turning point (the particle approaches, slows, reverses, and escapes) or no turning points (the particle passes through with nonzero speed everywhere).

In Figure 8.5.1, at the energy E shown, the particle is bound: it oscillates between x_1 and x_2, passing over the central barrier. If E were lowered below the barrier height at $x \approx 2.25$, the particle would be trapped in one of the two wells, unable to cross.

Small Oscillations About a Stable Equilibrium

Near a stable equilibrium at $x = x_0$, we can Taylor-expand $U(x)$:

$$U(x) \approx U(x_0) + \underbrace{U'(x_0)}_{=0}(x - x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2 + \cdots \tag{8.16}$$

The first-derivative term vanishes because x_0 is an equilibrium point. Defining the displacement $\xi = x - x_0$ and the effective spring constant $k_{\text{eff}} = U''(x_0)$:

$$U \approx U_0 + \tfrac{1}{2}k_{\text{eff}}\,\xi^2.$$

The corresponding force is $F = -\mathrm{d}U/\mathrm{d}\xi = -k_{\text{eff}}\,\xi$, which is Hooke's law. The equation of motion is:

$$m\ddot{\xi} = -k_{\text{eff}}\,\xi \quad \Longrightarrow \quad \omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{U''(x_0)}{m}}. \tag{8.17}$$

Key Point 8.5: All Small Oscillations Are Simple Harmonic

Near *any* stable equilibrium, a potential energy function is approximately parabolic. Therefore, all small oscillations about a stable equilibrium are approximately simple harmonic, with angular frequency $\omega = \sqrt{U''(x_0)/m}$. This is why simple harmonic motion appears so ubiquitously in physics: it is the universal behavior of any system near a stable equilibrium.

Worked Examples with Energy Diagrams

Example 8.13 (Stability analysis of a cubic-quadratic potential). A particle moves in the potential $U(x) = U_0\big[(x/a)^2 - 2(x/a)^3\big]$ with $U_0, a > 0$. (a) Find and classify the equilibrium positions. (b) Find the small-oscillation frequency near the stable equilibrium. (c) What minimum kinetic energy at $x = 0$ allows escape to $x \to +\infty$?

Solution. (a) Equilibria occur where $F = -\mathrm{d}U/\mathrm{d}x = 0$:

$$U'(x) = U_0\left[\frac{2x}{a^2} - \frac{6x^2}{a^3}\right] = \frac{2U_0 x}{a^2}\left[1 - \frac{3x}{a}\right] = 0.$$

This gives $x = 0$ and $x = a/3$. To classify:

$$U''(x) = U_0\left[\frac{2}{a^2} - \frac{12x}{a^3}\right].$$

At $x = 0$: $U''(0) = 2U_0/a^2 > 0 \Longrightarrow$ **stable**.

At $x = a/3$: $U''(a/3) = U_0(2/a^2 - 4/a^2) = -2U_0/a^2 < 0 \Longrightarrow$ **unstable**.

(b) Near $x = 0$, the effective spring constant is $k_{\text{eff}} = U''(0) = 2U_0/a^2$. The small-oscillation frequency is:

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{2U_0}{ma^2}}.$$

(c) The barrier is at $x = a/3$, where $U(a/3) = U_0(1/9 - 2/27) = U_0/27$. At $x = 0$, $U(0) = 0$. For the particle to escape over the barrier:

$$K_0 + U(0) \geq U(a/3) \quad \Longrightarrow \quad K_{\min} = \frac{U_0}{27}.$$

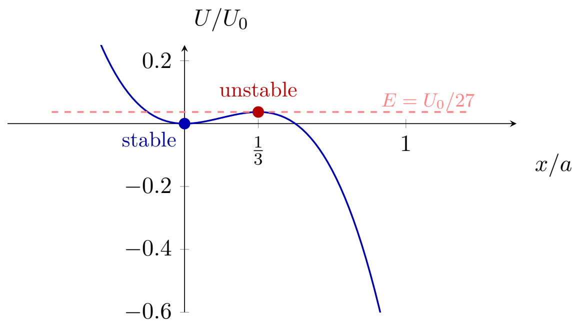


Figure 8.5.2: The potential $U(x) = U_0[(x/a)^2 - 2(x/a)^3]$ with a stable minimum at $x = 0$ and an unstable maximum at $x = a/3$. A particle at $x = 0$ needs kinetic energy $K \geq U_0/27$ to escape to $x \to +\infty$.

Example 8.14 (The Lennard-Jones potential). The interaction between two neutral atoms is often modeled by the Lennard-Jones potential:

$$U(r) = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right],$$

where ϵ is the depth of the potential well and σ is the distance at which $U = 0$. (a) Find the equilibrium separation r_0. (b) Find the binding energy (the energy needed to separate the atoms from equilibrium to $r \to \infty$). (c) Find the effective spring constant for small oscillations about r_0.

Solution. (a) Setting $\mathrm{d}U/\mathrm{d}r = 0$:

$$\frac{\mathrm{d}U}{\mathrm{d}r} = 4\epsilon\left[-\frac{12\sigma^{12}}{r^{13}} + \frac{6\sigma^6}{r^7}\right] = 0 \implies \frac{12\sigma^{12}}{r^{13}} = \frac{6\sigma^6}{r^7} \implies r_0^6 = 2\sigma^6 \implies r_0 = 2^{1/6}\sigma \approx 1.122\,\sigma.$$

(b) At equilibrium: $U(r_0) = 4\epsilon\left[\frac{1}{4} - \frac{1}{2}\right] = -\epsilon$. At infinity: $U(\infty) = 0$. The binding energy is:

$$\Delta U = 0 - (-\epsilon) = \epsilon.$$

(c) The effective spring constant is $k_{\text{eff}} = U''(r_0)$. Computing:

$$U''(r) = 4\epsilon\left[\frac{12 \cdot 13\,\sigma^{12}}{r^{14}} - \frac{6 \cdot 7\,\sigma^6}{r^8}\right].$$

At $r = r_0 = 2^{1/6}\sigma$, after substitution and simplification:

$$k_{\text{eff}} = U''(r_0) = \frac{72\epsilon}{\sigma^2} \cdot 2^{-1/3} = \frac{36 \cdot 2^{2/3}\,\epsilon}{\sigma^2} \approx \frac{57.1\,\epsilon}{\sigma^2}.$$

The small-oscillation angular frequency is $\omega = \sqrt{k_{\text{eff}}/\mu}$, where μ is the reduced mass of the two-atom system.

8.6 Energy Methods for Variable-Configuration Systems

Energy conservation extends naturally to systems whose geometry changes continuously, a rope sliding off a table, a chain piling up on a surface, a slinky uncoiling. The key technique is to compute the potential energy by *integrating* over all mass elements.

Rope Sliding off a Frictionless Table

Consider a uniform rope of mass m and length L, with a length y hanging over the edge of a frictionless table. Taking the table surface as the reference level ($U = 0$), the potential energy of the hanging segment is found by integrating:

$$U_{\text{hang}} = -\int_0^y \frac{m}{L}\, g\, \xi\, \mathrm{d}\xi = -\frac{mg}{2L}\, y^2,$$

where ξ is the distance below the table edge for each infinitesimal element $\mathrm{d}m = (m/L)\,\mathrm{d}\xi$, and the minus sign reflects that the hanging portion is *below* the reference level. The portion on the table has $U = 0$.

If the rope starts from rest with length y_0 hanging, energy conservation gives:

$$0 + \left(-\frac{mg}{2L} y_0^2\right) = \frac{1}{2} m v^2 + \left(-\frac{mg}{2L} y^2\right),$$

$$\boxed{v = \sqrt{\frac{g}{L}(y^2 - y_0^2)}.} \tag{8.18}$$

When the trailing end reaches the edge ($y = L$), the speed is $v = \sqrt{g(L^2 - y_0^2)/L}$.

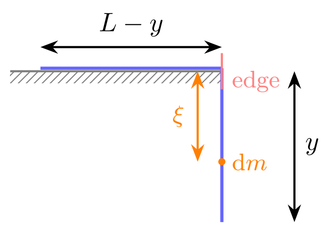


Figure 8.6.1: A rope sliding off a frictionless table. The variable ξ measures the distance below the edge for each mass element in the hanging portion.

Rope with Friction

When friction is present between the rope and table (coefficient μ_k), the portion lying flat (length $L - y$) experiences a normal force $(m/L)(L - y)\,g$ and contributes friction work. As the rope slides an infinitesimal distance $\mathrm{d}y$, the friction work is:

$$\mathrm{d}W_f = -\mu_k \frac{m}{L}(L - y)\, g\, \mathrm{d}y.$$

Integrating from y_0 to y:

$$W_f = -\frac{\mu_k m g}{L} \int_{y_0}^{y} (L - y')\, \mathrm{d}y' = -\frac{\mu_k m g}{L} \left[L y' - \frac{y'^2}{2} \right]_{y_0}^{y}.$$

The generalized work-energy theorem then gives:

$$W_f = \left(\tfrac{1}{2} m v^2 - \tfrac{mg}{2L} y^2\right) - \left(0 - \tfrac{mg}{2L} y_0^2\right),$$

which can be solved for $v(y)$. For $y_0 = L/4$ and $y = L$ (entire rope hanging), the result is:

$$v = \sqrt{\frac{gL(15 - 9\mu_k)}{16}}.$$

The rope slides only if $v^2 > 0$, i.e., $\mu_k < 5/3$.

Equation of Motion via Energy Methods

The energy approach also yields the equation of motion. Differentiating $E = \frac{1}{2}mv^2 - \frac{mg}{2L}y^2 = \text{const}$ with respect to time (using $v = \dot{y}$):

$$m\dot{y}\,\ddot{y} - \frac{mg}{L}\,y\,\dot{y} = 0 \quad \Longrightarrow \quad \ddot{y} = \frac{g}{L}\,y.$$

This is a second-order ODE with exponentially growing solutions: $y(t) = A\,e^{t\sqrt{g/L}} + B\,e^{-t\sqrt{g/L}}$. The rope accelerates as more mass hangs over the edge, a positive feedback process. This is in contrast to the constant-acceleration problems of earlier chapters and provides a good example of how energy methods can be used to derive equations of motion.

Worked Example: A Falling Chain

Example 8.15 (Chain piling on a scale). A uniform chain of mass m and length L is held vertically with its lower end just touching a scale. The chain is released and falls freely. Find the force on the scale as a function of the length y that has piled up.

Solution. At the instant a length y has piled up, the remaining length $L - y$ is still falling. The center of mass of the still-falling portion is at height $(L - y)/2$ above the scale.

By energy conservation, each link arrives at the scale with speed $v = \sqrt{2gy'}$, where y' is the distance it has fallen. Since the chain was released with the bottom touching the scale, a link at height y' above the pile has fallen y', so $v = \sqrt{2gy}$ (the link that is just arriving has fallen a distance y).

The scale must support two contributions:

1. The **weight** of the piled portion: $W_{\text{pile}} = (m/L)\,y\,g$.
2. The **impulse force** from stopping the arriving links. In time $\mathrm{d}t$, a mass $\mathrm{d}m = (m/L)\,v\,\mathrm{d}t$ arrives with momentum $\mathrm{d}m \cdot v$, which must be reduced to zero. The force is:

$$F_{\text{impulse}} = \frac{\mathrm{d}m}{\mathrm{d}t}\,v = \frac{m}{L}\,v^2 = \frac{m}{L}(2gy) = \frac{2mgy}{L}.$$

Total force on the scale:

$$\boxed{F(y) = \frac{mgy}{L} + \frac{2mgy}{L} = \frac{3mgy}{L}.}$$

When the entire chain has piled up ($y = L$), $F = 3mg$—three times the weight. This surprising factor of 3 comes from the fact that the impulse force contributes twice the static weight at each instant.

8.7 Energy Methods: A Problem-Solving Summary

We have now developed three equivalent formulations of the energy principle. For reference, here is a compact summary of when to use each.

Strategy 8.3: Choosing the Right Energy Method

1. **Work-Energy Theorem** ($W_{\text{net}} = \Delta K$): Use when you want to track the work done by *every* force explicitly. No potential energy is used. Best for problems where forces are known but potential energies have not been defined.

2. **Conservation of Mechanical Energy** ($K_i + U_i = K_f + U_f$): Use when *only* conservative forces do work. Simplest approach: no forces need to be computed, just compare energies at two points. Cannot be used if friction or other non-conservative forces are present.

3. **Generalized Work-Energy Theorem** ($W_{\text{nc}} = \Delta K + \Delta U$): Use when *both* conservative and non-conservative forces act. Conservative forces are handled through U; only the non-conservative work appears on the left. This is the most general and most commonly used form.

All three are equivalent. The choice is a matter of convenience: use the simplest method that fits the problem. When in doubt, the generalized form (Form 3) always works.

Common Mistake 8.2: Energy Methods Give Speed, Not Direction

All energy methods yield the *speed* at a given point, not the velocity vector. They cannot tell you which direction the object is moving. If the direction of motion matters, you need additional information—typically from the problem setup, conservation of momentum, or Newton's second law.

Problems

Problem 8.1 ⋆

A 2.0 kg ball is thrown vertically upward at 15 m/s. Using energy conservation (not kinematics), find the maximum height. Take $g = 10\,\mathrm{m/s^2}$.

Problem 8.2 ⋆

A spring ($k = 800\,\mathrm{N/m}$) is compressed 0.05 m from its natural length. How much potential energy is stored? If a 0.20 kg ball is placed against the spring and released on a frictionless surface, what speed does the ball reach?

Problem 8.3 ⋆⋆

A roller-coaster car of mass m starts from rest at height h_1 and passes over a second hill of height $h_2 < h_1$ (frictionless track). (a) Find the speed at the top of the second hill. (b) What is the maximum height h_2 that the car can clear?

Problem 8.4 ⋆⋆

A 0.25 kg ball is dropped from 2.0 m. It bounces back to 1.5 m. (a) How much mechanical energy was lost? (b) What fraction of the kinetic energy just before impact was lost during the bounce? (c) Where did the lost energy go?

Problem 8.5 ⋆⋆

A spring gun on a table of height H fires a ball of mass m horizontally. The spring (constant k) was compressed by Δx. Find where the ball lands on the floor (horizontal distance from the table edge).

Problem 8.6 ⋆⋆

A block slides down a frictionless ramp of height h and around a vertical loop of radius R. (a) Derive $h_{\min}$ for the block to complete the loop. (b) For $h = 3R$, find the normal force at the top in terms of mg. (c) Find the normal force at the bottom of the loop for $h = 3R$.

Problem 8.7 ⋆⋆⋆

A pendulum of length ℓ and mass m is released from angle θ_0. (a) Derive the speed at the bottom. (b) Derive the tension at the bottom. (c) Derive the tension at a general angle $\theta < \theta_0$. (d) For what release angle θ_0 does the tension at the bottom equal $3mg$?

Problem 8.8 ⋆⋆⋆

A ball of mass m is attached to a string of length ℓ and swings as a pendulum. (a) What minimum speed at the bottom is needed for the ball to make a complete vertical circle? (b) What is the tension at the top of the circle in this limiting case? (c) How does this result change if the string is replaced by a rigid rod?

Problem 8.9 ⋆⋆⋆

A block of mass m slides from rest down a frictionless ramp of height h onto a rough horizontal surface (μ_k). (a) Find the stopping distance. (b) If the block instead starts with speed v_0 at the top of the ramp, find the stopping distance on the horizontal surface.

Problem 8.10 ⋆⋆⋆

A 50 kg skier starts from rest at the top of a 30 m slope inclined at 25° with $\mu_k = 0.10$. (a) Find the speed at the bottom. (b) The skier then encounters a flat section with $\mu_k = 0.15$. How far does the skier travel before stopping? (c) What fraction of the initial potential energy was converted to thermal energy?

Problem 8.11 ⋆⋆⋆

Verify that the force $\boldsymbol{F} = (2xy + z^2)\,\hat{\boldsymbol{\imath}} + x^2\,\hat{\boldsymbol{\jmath}} + 2xz\,\hat{\boldsymbol{k}}$ is conservative by computing $\nabla \times \boldsymbol{F}$. Then find the potential energy function $U(x, y, z)$ such that $\boldsymbol{F} = -\nabla U$.

Problem 8.12 ⋆⋆⋆⋆

A particle moves under the potential $U(x) = U_0\big[(x/a)^2 - 2(x/a)^3\big]$ with $U_0, a > 0$. (a) Find the equilibrium positions and classify them. (b) Find $F(x)$. (c) A particle starts from rest at $x = a$. Describe the subsequent motion qualitatively. (d) What minimum kinetic energy at $x = 0$ allows escape to $x \to +\infty$? (e) Find the frequency of small oscillations about the stable equilibrium.

Problem 8.13 ⋆⋆⋆⋆

A uniform rope of mass m and length L hangs with $\frac{1}{4}$ of its length over the edge of a frictionless table. (a) Find the speed when the trailing end reaches the edge. (b) Repeat with kinetic friction μ_k between the rope and table. (c) For what value of μ_k does the rope not slide at all?

Problem 8.14 ⋆⋆⋆⋆

A particle moves in the potential $U(x) = U_0 \cosh(x/a)$ where $\cosh(z) = (e^z + e^{-z})/2$. (a) Find the equilibrium position and show it is stable. (b) Show that small oscillations about equilibrium are SHM with $\omega = \sqrt{U_0/(ma^2)}$. (c) Sketch $U(x)$ and describe the motion for total energy $E > U_0$. Is the motion bound or unbound?

Problem 8.15 ⋆⋆⋆⋆

A bead of mass m slides without friction on a wire bent into the shape $y = bx^4$, where b is a positive constant. (a) Find the potential energy $U(x)$ (take $U = 0$ at $x = 0$). (b) Show that $x = 0$ is a stable equilibrium. (c) Are small oscillations about $x = 0$ simple harmonic? If not, why does the standard Taylor expansion argument fail, and what is the leading-order restoring force?

Problem 8.16 ⋆⋆⋆⋆⋆

(*Escape velocity from energy conservation.*) A projectile of mass m is launched vertically from the surface of a planet of mass M and radius R. Using conservation of energy with the universal gravitational potential $U = -GMm/r$: (a) derive the escape speed $v_e = \sqrt{2GM/R}$; (b) show that for $v_0 \ll v_e$, the maximum height reduces to the flat-Earth result $h = v_0^2/(2g)$ where $g = GM/R^2$; (c) find the exact maximum height for arbitrary $v_0 < v_e$.

Problem 8.17 ⋆⋆⋆⋆⋆

(*Proving path independence from the curl condition.*) Consider the force $\boldsymbol{F} = (y^2z^3)\,\hat{\boldsymbol{\imath}} + (2xyz^3)\,\hat{\boldsymbol{\jmath}} + (3xy^2z^2)\,\hat{\boldsymbol{k}}$. (a) Show that $\nabla \times \boldsymbol{F} = \boldsymbol{0}$. (b) Find $U(x, y, z)$ by integrating F_x with respect to x, then determining the "constants" of integration by matching F_y and F_z. (c) Verify by computing $-\nabla U$ and recovering $\boldsymbol{F}$.

Chapter 9

Linear Momentum and Impulse

In the preceding chapters we developed two powerful frameworks for analyzing motion: Newton's laws (force and acceleration) and energy methods (work, kinetic energy, potential energy). Each has strengths: Newton's laws give the full trajectory but require tracking vector forces at every instant; energy methods bypass forces but yield only speeds, not directions, and give no information about time.

In this chapter we introduce a third conserved quantity (**linear momentum**) and develop the **impulse-momentum theorem**, which relates forces acting over time to changes in momentum. The crown jewel is **conservation of momentum**: when no external forces act on a system, its total momentum is constant. This principle is indispensable for analyzing **collisions**, **explosions**, **recoil**, and **rocket propulsion**—problems where forces are unknown, brief, or internal to the system.

Together, the three conservation laws (energy, momentum, and later angular momentum) form the backbone of all of mechanics.

9.1 Linear Momentum

Definition and Basic Properties

Definition 9.1: Linear Momentum

The **linear momentum** of a particle of mass m and velocity $\boldsymbol{v}$ is

$$\boxed{\boldsymbol{p} = m\boldsymbol{v}.} \tag{9.1}$$

Momentum is a vector quantity with the same direction as the velocity. Its SI unit is kg m/s, which is equivalent to N s.

Several features deserve emphasis:

- **Vector nature:** Unlike kinetic energy (a scalar), momentum has both magnitude and direction. Two objects of the same mass moving at the same speed in opposite directions have the same kinetic energy but *opposite* momenta.
- **Linear in velocity:** While $K \propto v^2$, momentum is $p \propto v$. Doubling the speed doubles the momentum but quadruples the kinetic energy.
- **Additive:** The total momentum of a system is the vector sum of the momenta of all its parts: $\boldsymbol{p}_{\text{total}} = \sum_i m_i \boldsymbol{v}_i$.
- **Component form:** $p_x = mv_x$, $p_y = mv_y$, $p_z = mv_z$. Each component is independently conserved when the corresponding component of external force vanishes.

Momentum and Kinetic Energy: Two Measures of Motion

Momentum and kinetic energy both measure "how much motion" an object has, but they do so in fundamentally different ways:

$$\boldsymbol{p} = m\boldsymbol{v} \quad \text{(vector, linear in } v\text{)}, \qquad K = \tfrac{1}{2}mv^2 \quad \text{(scalar, quadratic in } v\text{)}. \tag{9.2}$$

Neither quantity is "more fundamental" than the other—each captures information the other misses. The relationship between them is:

$$K = \frac{p^2}{2m}, \tag{9.3}$$

which follows directly from $p = mv$ and $K = \frac{1}{2}mv^2$. This is often useful for converting between the two: given the momentum, you can immediately find the kinetic energy without knowing v and m separately.

Example 9.1 (Comparing a truck and a bullet). A 10 000 kg truck moves at 2 m/s; a 0.010 kg bullet moves at 500 m/s. Compare their momenta and kinetic energies.

Solution. Truck: $p = 10000 \times 2 = 20\,000\,\text{kg m/s}$, $K = \frac{1}{2}(10000)(4) = 20\,000\,\text{J}$.

Bullet: $p = 0.010 \times 500 = 5\,\text{kg m/s}$, $K = \frac{1}{2}(0.010)(250000) = 1250\,\text{J}$.

The truck has 4000 times the momentum of the bullet but only 16 times the kinetic energy. Momentum scales linearly with speed, while energy scales quadratically, so the slow, heavy truck dominates in momentum, but the fast, light bullet is more energy-efficient per unit mass.

Newton's Second Law in Momentum Form

Newton originally stated his second law not as $\boldsymbol{F} = m\boldsymbol{a}$, but as:

$$\boxed{\boldsymbol{F}_{\text{net}} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}.} \tag{9.4}$$

The net force on an object equals the *rate of change of its momentum*. For constant mass, $\mathrm{d}\boldsymbol{p}/\mathrm{d}t = m\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t = m\boldsymbol{a}$, recovering the familiar $\boldsymbol{F}_{\text{net}} = m\boldsymbol{a}$.

The momentum form (9.4) is more general because it also applies when the mass changes: for example, a rocket burning fuel, a raindrop collecting moisture, or sand falling onto a moving conveyor belt. We will exploit this generality in Section 9.6.

Key Point 9.1: When to Use the Momentum Form of $N2$

Use $\boldsymbol{F}_{\text{net}} = m\boldsymbol{a}$ when the mass is constant. Use $\boldsymbol{F}_{\text{net}} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$ when the mass is changing, when the force acts over a known time interval, or when you want to relate force to change in momentum directly.

Momentum of a System of Particles

For a system of N particles, the total momentum is:

$$\boldsymbol{P} = \sum_{i=1}^{N} m_i \boldsymbol{v}_i = M\boldsymbol{v}_{\text{cm}}, \tag{9.5}$$

where $M = \sum m_i$ is the total mass and $\boldsymbol{v}_{\text{cm}}$ is the velocity of the center of mass. This important result—that the total momentum equals the total mass times the center-of-mass velocity—connects momentum to the center-of-mass concept developed later in this chapter (Section 9.8).

Differentiating:

$$\boldsymbol{F}_{\text{ext}} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = M\boldsymbol{a}_{\text{cm}}. \tag{9.6}$$

The center of mass of a system moves as if all the mass were concentrated there and all external forces acted on that point. Internal forces cancel in pairs (Newton's third law) and do not affect the total momentum.

9.2 Impulse and the Impulse-Momentum Theorem

Definition of Impulse

Definition 9.2: Impulse

The **impulse** delivered by a force $\boldsymbol{F}$ acting over the time interval $[t_i, t_f]$ is

$$\boxed{\boldsymbol{J} = \int_{t_i}^{t_f} \boldsymbol{F}\,\mathrm{d}t.} \tag{9.7}$$

For a constant force: $\boldsymbol{J} = \boldsymbol{F}\,\Delta t$.

Impulse has units of N s = kg m/s—the same units as momentum. This is no coincidence: the impulse-momentum theorem states that impulse *equals* the change in momentum.

The Impulse-Momentum Theorem

Theorem 9.1: Impulse-Momentum Theorem

The impulse delivered to an object equals the change in its momentum:

$$\boxed{\boldsymbol{J} = \Delta\boldsymbol{p} = \boldsymbol{p}_f - \boldsymbol{p}_i.} \tag{9.8}$$

Derivation. Starting from Newton's second law in momentum form:

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \quad\Longrightarrow\quad \boldsymbol{F}\,\mathrm{d}t = \mathrm{d}\boldsymbol{p}.$$

Integrating both sides from t_i to t_f:

$$\int_{t_i}^{t_f} \boldsymbol{F}\,\mathrm{d}t = \int_{\boldsymbol{p}_i}^{\boldsymbol{p}_f} \mathrm{d}\boldsymbol{p} = \boldsymbol{p}_f - \boldsymbol{p}_i = \Delta\boldsymbol{p}.$$

The impulse-momentum theorem is completely analogous to the work-energy theorem:

Theorem	Relates	Involves
Work-energy	Force × distance → ΔK	position integral
Impulse-momentum	Force × time → $\Delta\boldsymbol{p}$	time integral

The work-energy theorem tells us how much the speed changes; the impulse-momentum theorem tells us how much the velocity (magnitude *and direction*) changes.

Average Force

When the force varies in time (as in a collision), we define the **average force**:

$$\boldsymbol{F}_{\text{avg}} = \frac{\boldsymbol{J}}{\Delta t} = \frac{\Delta \boldsymbol{p}}{\Delta t}. \tag{9.9}$$

The average force produces the same impulse over the same time interval as the actual time-varying force.

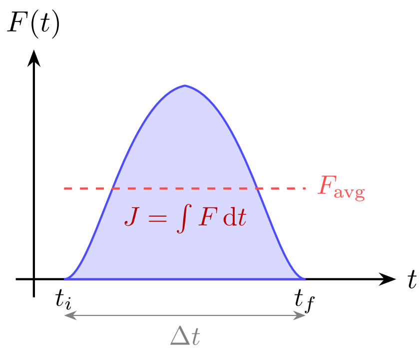


Figure 9.2.1: The impulse $J = \int F \, \mathrm{d}t$ equals the area under the $F(t)$ curve. The average force F_{avg} (dashed) produces the same impulse (same area) over the same interval Δt.

The Physics of Collisions: Hard vs. Soft

The impulse-momentum theorem has a crucial practical implication: for a given change in momentum $\Delta \boldsymbol{p}$, the average force is inversely proportional to the collision time Δt:

$$F_{\text{avg}} = \frac{|\Delta p|}{\Delta t}.$$

A longer collision time means a smaller force. This is the physical principle behind every safety device:

- **Airbags** increase the time over which a passenger decelerates from perhaps 5 ms (hitting the dashboard) to 50 ms, reducing the average force by a factor of 10.
- **Crumple zones** in cars deform during impact, extending the collision time.
- **Bending your knees** when landing from a jump increases the deceleration time.
- **A baseball glove** recoils during a catch, extending the time over which the ball's momentum is absorbed.

Key Point 9.2: Reducing Impact Force

For a given impulse $J = \Delta p$ (fixed by the physics of the collision), the average force satisfies $F_{\text{avg}} = J/\Delta t$. *Increasing the collision time always decreases the average force.* This principle underlies all safety engineering.

Worked Examples

Example 9.2 (Baseball hit). A 0.145 kg baseball arrives at 40 m/s and is hit directly back at 50 m/s. The bat is in contact with the ball for 1.0 ms. Find the impulse and average force.

Solution. Taking the initial direction as positive:

$$J = m(v_f - v_i) = 0.145(-50 - 40) = 0.145 \times (-90) = -13.05\,\mathrm{N\,s}.$$

The magnitude is 13.05 N s, directed back toward the pitcher. The average force:

$$F_{\text{avg}} = \frac{|J|}{\Delta t} = \frac{13.05}{0.001} = 13\,050\,\mathrm{N} \approx 13\,\mathrm{kN}.$$

This is roughly 9200 times the ball's weight, an enormous force, but it acts for only a millisecond.

Example 9.3 (Wall bounce). A 0.40 kg ball traveling at 12 m/s strikes a wall and rebounds at 9.0 m/s. Contact time is 0.020 s. Find (a) the impulse and (b) the average force.

Solution. Taking rightward (toward the wall) as positive:

(a) $J = m(v_f - v_i) = 0.40(-9.0 - 12) = 0.40(-21) = -8.4\,\mathrm{N\,s}$. The impulse is 8.4 N s directed away from the wall.

(b) $F_{\text{avg}} = J/\Delta t = -8.4/0.020 = -420\,\mathrm{N}$ (420 N away from the wall).

Note that the impulse (and average force) for a ball that *bounces* is larger than for one that *sticks*. A ball that sticks has $\Delta p = -mv_i$; a ball that bounces has $\Delta p = -m(v_i + v_f)$, which is larger in magnitude.

Example 9.4 (Exponentially decaying force). A force $F(t) = F_0 e^{-t/\tau}$ acts on a particle of mass m initially at rest. Find (a) the total impulse, (b) the final speed, and (c) the speed at time t.

Solution. (a) $J = \int_0^\infty F_0 e^{-t/\tau}\,\mathrm{d}t = F_0[-\tau e^{-t/\tau}]_0^\infty = F_0\tau$.

(b) $v_\infty = J/m = F_0\tau/m$.

(c) $v(t) = \frac{1}{m}\int_0^t F_0 e^{-t'/\tau}\,\mathrm{d}t' = \frac{F_0\tau}{m}(1 - e^{-t/\tau})$.

The speed approaches v_∞ exponentially with time constant τ.

Example 9.5 (Car crash: the importance of collision time). A 1500 kg car traveling at 20 m/s ($\approx$ 72 km/h) hits a wall and stops. Compare the average force when (a) the car has a rigid frame ($\Delta t \approx 0.05\,\mathrm{s}$) and (b) the car has a crumple zone ($\Delta t \approx 0.50\,\mathrm{s}$).

Solution. The impulse is the same in both cases: $|J| = m|v_f - v_i| = 1500(20) = 30\,000\,\mathrm{N\,s}$.

(a) Rigid: $F_{\text{avg}} = 30000/0.05 = 600\,000\,\mathrm{N} = 600\,\mathrm{kN}$ (about 41 times the car's weight).

(b) Crumple zone: $F_{\text{avg}} = 30000/0.50 = 60\,000\,\mathrm{N} = 60\,\mathrm{kN}$ (about 4.1 times the car's weight).

The crumple zone reduces the average force by a factor of 10. This is a life-saving difference.

9.3 Conservation of Linear Momentum

Derivation from Newton's Third Law

Conservation of momentum follows directly from Newton's second and third laws. Consider a system of two particles. By Newton's second law applied to the system:

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{p}_1}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{p}_2}{\mathrm{d}t} = \boldsymbol{F}_1^{\text{ext}} + \boldsymbol{F}_{12} + \boldsymbol{F}_2^{\text{ext}} + \boldsymbol{F}_{21}.$$

By Newton's third law, $\boldsymbol{F}_{12} + \boldsymbol{F}_{21} = \boldsymbol{0}$. Therefore:

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \boldsymbol{F}_1^{\text{ext}} + \boldsymbol{F}_2^{\text{ext}} = \boldsymbol{F}_{\text{total}}^{\text{ext}}.$$

If the total external force vanishes ($\boldsymbol{F}_{\text{total}}^{\text{ext}} = \boldsymbol{0}$), then $\mathrm{d}\boldsymbol{P}/\mathrm{d}t = \boldsymbol{0}$, so $\boldsymbol{P}$ is constant. The argument generalizes immediately to N particles: all internal forces cancel in pairs, leaving only the net external force.

Theorem 9.2: Conservation of Linear Momentum

If the net external force on a system is zero, the total momentum of the system is conserved:

$$\boxed{\boldsymbol{F}_{\text{total}}^{\text{ext}} = \mathbf{0} \implies \sum_i \boldsymbol{p}_i = \text{constant.}} \tag{9.10}$$

Momentum conservation holds *component by component*: if $F_x^{\text{ext}} = 0$, then p_x is conserved even if p_y and p_z are not.

Common Mistake 9.1: Internal Forces Do Not Change Total Momentum

Internal forces—no matter how large—*never* change the total momentum of a system. During a collision, the forces between colliding objects can be enormous, but they are action–reaction pairs and cancel in the sum. This is why momentum is conserved during collisions, explosions, and all other internal interactions.

Momentum Conservation in Collisions

During a collision, the internal impulsive forces between colliding objects vastly exceed external forces (gravity, friction) over the short collision time Δt. Therefore, to excellent approximation:

$$\sum \boldsymbol{p}_{\text{before}} \approx \sum \boldsymbol{p}_{\text{after}} \qquad \text{(during any collision).} \tag{9.11}$$

This holds regardless of whether the collision is elastic, inelastic, or perfectly inelastic. Momentum conservation during collisions is one of the most widely used principles in physics.

Key Point 9.3: When Is Momentum Conserved?

Momentum conservation applies whenever the net external force is zero (or negligible compared to internal forces, as in a brief collision). Common scenarios:

- **Collisions** (any type): internal forces dominate during the brief impact.
- **Explosions**: the explosion forces are internal to the system of fragments.
- **Recoil**: gun + bullet, cannon + cannonball, astronaut + tool.
- **Component-wise**: if external forces act only in one direction (e.g., gravity acts vertically), then horizontal momentum is conserved even though vertical momentum is not.

Worked Examples

Example 9.6 (Recoil of a cannon). A cannon of mass M fires a cannonball of mass m at speed v_0. If the cannon is initially at rest and free to roll on a frictionless surface, find the recoil speed.

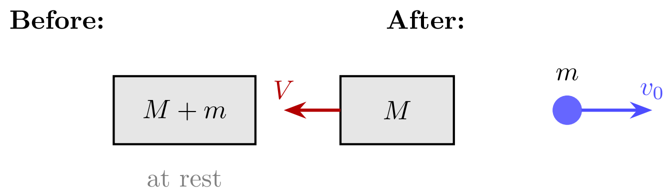


Figure 9.3.1: A cannon at rest fires a cannonball. The total initial momentum is zero, so the cannon recoils to conserve momentum.

Solution. Initial momentum: $\boldsymbol{P}_i = 0$ (system at rest). After firing, momentum conservation:

$$0 = mv_0 + MV \quad \Longrightarrow \quad V = -\frac{m}{M}v_0.$$

The minus sign indicates the cannon recoils in the direction opposite to the cannonball. If $m/M = 1/100$, the recoil speed is only 1% of the cannonball's speed, but the kinetic energies are not equal: $K_{\text{cannon}}/K_{\text{ball}} = (MV^2)/(mv_0^2) = m/M = 1/100$. Most of the energy goes to the cannonball.

Example 9.7 (Astronaut in space). A 70 kg astronaut floating at rest in space throws a 5.0 kg tool kit at 6.0 m/s relative to the spaceship. Find the astronaut's recoil speed.

Solution. $0 = m_t v_t + m_a v_a$. $v_a = -m_t v_t/m_a = -5.0(6.0)/70 = -0.43\,\text{m/s}$.

The astronaut drifts at 0.43 m/s in the direction opposite to the throw. To return to the ship, the astronaut would need to throw something *away* from the ship.

Example 9.8 (Explosion into three pieces). A 10 kg object at rest explodes into three pieces: a 3 kg piece flies north at 8 m/s, a 4 kg piece flies east at 6 m/s. Find the velocity of the third piece.

Solution. The third piece has mass $m_3 = 10 - 3 - 4 = 3\,\text{kg}$. Momentum conservation component by component ($\boldsymbol{P}_i = \boldsymbol{0}$):

$$x: \quad 0 = 0 + 4(6) + 3v_{3x} \quad \Longrightarrow \quad v_{3x} = -8\,\text{m/s},$$
$$y: \quad 0 = 3(8) + 0 + 3v_{3y} \quad \Longrightarrow \quad v_{3y} = -8\,\text{m/s}.$$

The third piece moves at $v_3 = \sqrt{8^2 + 8^2} = 8\sqrt{2} \approx 11.3\,\text{m/s}$ in the direction 45° south of west.

Example 9.9 (Component-wise conservation). A 2.0 kg ball is thrown horizontally at 10 m/s from a height of 5.0 m. Just before hitting the ground, what are the horizontal and vertical components of momentum?

Solution. Horizontal: no horizontal forces act, so p_x is conserved: $p_x = mv_x = 2.0(10) = 20\,\text{kg m/s}$.

Vertical: gravity acts, so p_y is *not* conserved. Using kinematics: $v_y = \sqrt{2gh} = \sqrt{2(9.8)(5.0)} = 9.9\,\text{m/s}$ (downward). So $p_y = mv_y = 2.0(9.9) = 19.8\,\text{kg m/s}$ downward.

9.4 Applications: Multi-Phase Problems

Many problems involve two or more distinct phases that require different physical principles. A common pattern is:

1. A **collision phase** (brief, large internal forces): use *momentum conservation.*
2. A **before/after phase** (extended, known forces): use *kinematics, energy conservation, or Newton's laws.*

The key is to apply the correct physics to each phase *separately* and connect them at the transition point.

Strategy 9.1: Multi-Phase Problem Solving

1. Identify the phases: which involve collisions (momentum conservation) and which involve forces/motion (Newton's laws, energy, kinematics)?
2. Apply the appropriate principle to each phase independently.
3. Connect phases through shared quantities (the final state of one phase is the initial state of the next).
4. Remember: energy is generally *not* conserved during inelastic collisions, but momentum is. After the collision, energy conservation may apply (if forces are conservative).

Worked Examples

Example 9.10 (Ballistic pendulum). A bullet of mass m traveling at speed v_0 embeds itself in a wooden block of mass M suspended as a pendulum of length ℓ. Find the height h to which the pendulum swings.

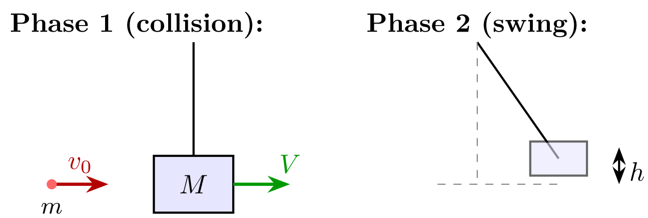


Figure 9.4.1: The ballistic pendulum. Phase 1: the bullet embeds in the block (perfectly inelastic collision; momentum conserved, energy not). Phase 2: the block+bullet swings upward (energy conserved).

Solution. Phase 1 (collision): Momentum conservation (energy is *not* conserved: the collision is perfectly inelastic):

$$mv_0 = (m+M)V \implies V = \frac{mv_0}{m+M}.$$

Phase 2 (swing): Energy conservation (no non-conservative forces after the collision):

$$\tfrac{1}{2}(m+M)V^2 = (m+M)gh \implies h = \frac{V^2}{2g} = \frac{m^2v_0^2}{2g(m+M)^2}.$$

To find v_0 from the measured height h:

$$\boxed{v_0 = \frac{m+M}{m}\sqrt{2gh}.}$$

This was historically used to measure bullet speeds before electronic timers existed.

Example 9.11 (Projectile + collision on a frozen lake). A ball of mass m is launched at speed V and angle θ from the surface of a frozen lake. It lands on a stationary sledge of mass M and sticks to it. Find (a) the final speed of the sledge and (b) the energy converted to other forms.

Solution. (a) The ball lands with the same speed V at which it was launched (same height, frictionless). Its horizontal component is $V\cos\theta$. The vertical component is absorbed by the ice surface (a normal force provides the vertical impulse). Horizontal momentum is conserved:

$$mV\cos\theta = (m+M)V_f \implies V_f = \frac{mV\cos\theta}{m+M}.$$

(b) $K_i = \frac{1}{2}mV^2$ (the ball's KE just before impact). $K_f = \frac{1}{2}(m+M)V_f^2$. Energy lost:

$$\Delta E = K_i - K_f = \frac{1}{2}mV^2 - \frac{1}{2}\frac{m^2V^2\cos^2\theta}{m+M} = \frac{1}{2}mV^2\left(1 - \frac{m\cos^2\theta}{m+M}\right) = \frac{1}{2}mV^2 \cdot \frac{M + m\sin^2\theta}{m+M}.$$

Example 9.12 (Explosion followed by projectile motion). A firework of mass $3m$ is launched vertically and reaches its peak height H. At the peak it explodes into three equal pieces. One piece is launched horizontally to the right at speed v_0; a second is launched horizontally to the left at speed v_0. What happens to the third piece?

Solution. At the peak, the firework is momentarily at rest: $\boldsymbol{P}_i = \mathbf{0}$. After the explosion:

$$x: \quad 0 = mv_0 + m(-v_0) + mv_{3x} \quad \Longrightarrow \quad v_{3x} = 0.$$

$$y: \quad 0 = 0 + 0 + mv_{3y} \quad \Longrightarrow \quad v_{3y} = 0.$$

The third piece has zero velocity: it is momentarily at rest at height H and simply falls straight down.

9.5 Continuous Mass Flow and Force

Many practical situations involve a *continuous stream* of matter striking, leaving, or passing through a system—water from a hose hitting a wall, sand landing on a conveyor belt, exhaust gas leaving a rocket. In these problems, momentum is transferred continuously, and the stream exerts a steady force.

Force from a Steady Stream

Consider a stream of particles, each of mass $\mathrm{d}m$, arriving at speed v and being brought to rest (or deflected) by a surface. In a time interval $\mathrm{d}t$, a mass $\mathrm{d}m = \dot{m}\,\mathrm{d}t$ arrives, where $\dot{m} = \mathrm{d}m/\mathrm{d}t$ is the **mass flow rate** (in kg/s). Each element carries momentum $v\,\mathrm{d}m$ and must be brought to rest, so the impulse required is:

$$\mathrm{d}J = v\,\mathrm{d}m = \dot{m}\,v\,\mathrm{d}t.$$

The force exerted *on the stream* (and by Newton's third law, the reaction force exerted *by the stream on the surface*) is:

$$F = \frac{\mathrm{d}J}{\mathrm{d}t} = \dot{m}\,v. \tag{9.12}$$

If the stream bounces back elastically (reversing its velocity), the momentum change per element doubles:

$$F_{\text{elastic}} = 2\dot{m}\,v. \tag{9.13}$$

More generally, if each element arrives with velocity v_i and leaves with velocity v_f, the force on the surface is:

$$F = \dot{m}(v_i - v_f). \tag{9.14}$$

Key Point 9.4: Force from a Steady Stream

A steady stream of matter hitting (or leaving) a surface exerts a force $F = \dot{m}\,\Delta v$, where $\dot{m}$ is the mass flow rate and Δv is the change in speed of each mass element. This is a direct application of the impulse-momentum theorem in the continuous limit.

Worked Examples

Example 9.13 (Fire hose against a wall). A fire hose delivers water at a rate $\dot{m} = 2.0\,\mathrm{kg/s}$ with speed $v = 20\,\mathrm{m/s}$ against a flat wall. Find the force on the wall (a) if the water does not bounce back and (b) if the water bounces back at the same speed.

Solution. (a) Inelastic ($v_f = 0$): $F = \dot{m}(v - 0) = 2.0(20) = 40\,\mathrm{N}$.

(b) Elastic ($v_f = -v$): $F = \dot{m}(v - (-v)) = \dot{m}(2v) = 2.0(40) = 80\,\mathrm{N}$.

The elastic case gives exactly twice the force, because the momentum change per unit mass is $2v$ instead of v.

Example 9.14 (Sand on a conveyor belt). Sand drops vertically onto a horizontal conveyor belt moving at constant speed v. Sand lands at a rate $\dot{m}$. Find the force required to keep the belt moving at constant speed.

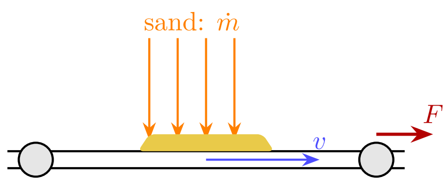


Figure 9.5.1: Sand dropping vertically onto a conveyor belt moving at constant speed v.

Solution. Each element of sand arrives with zero horizontal momentum and must be accelerated to belt speed v. In time $\mathrm{d}t$, mass $\mathrm{d}m = \dot{m}\,\mathrm{d}t$ is accelerated from 0 to v, requiring horizontal impulse $v\,\mathrm{d}m$. The force is:

$$F = \dot{m}\,v.$$

Note: this force does work at rate $P = Fv = \dot{m}v^2$. But the kinetic energy gained by the sand per unit time is only $\frac{1}{2}\dot{m}v^2$. The other half, $\frac{1}{2}\dot{m}v^2$, is dissipated as heat due to the sand sliding on the belt before it reaches belt speed. Energy is *not* conserved in this inelastic accretion process; momentum is the correct tool.

Example 9.15 (Sand car: accreting mass). An open railroad car of mass M rolls at speed v_0 on a frictionless track under a hopper that drops sand vertically into the car at rate $\dot{m}$. Find $v(t)$.

Solution. The sand has zero horizontal momentum before landing. No external horizontal forces act. Horizontal momentum is conserved:

$$p_x = Mv_0 = (M + \dot{m}\,t)\,v(t) \quad \Longrightarrow \quad \boxed{v(t) = \frac{Mv_0}{M + \dot{m}\,t}}.$$

The car slows down as it accumulates mass. As $t \to \infty$, $v \to 0$—all the initial kinetic energy is eventually dissipated as heat during the inelastic collection of sand.

The kinetic energy at time t is $K = \frac{1}{2}(M + \dot{m}t)v^2 = \frac{M^2v_0^2}{2(M+\dot{m}t)}$, which decreases monotonically. The "lost" kinetic energy goes into heating the sand (and car) through the inelastic collision as each sand grain is accelerated horizontally.

9.6 The Rocket Equation

Rockets present a unique challenge: the system's mass decreases as fuel is ejected. Neither the standard $F = ma$ (which assumes constant mass) nor simple momentum conservation (which assumes a fixed system) directly applies. We must use the momentum form $F = \mathrm{d}p/\mathrm{d}t$ applied to the *entire system* (rocket + ejected gas) over an infinitesimal time interval.

Derivation

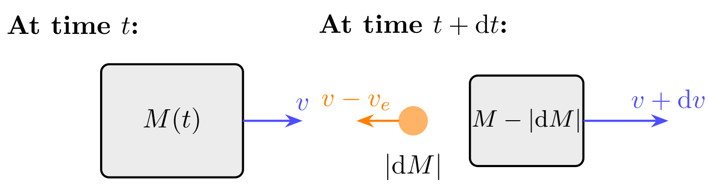


Figure 9.6.1: Rocket propulsion analyzed over an infinitesimal time interval. The rocket ejects mass $|\mathrm{d}M|$ at velocity $v - v_e$ in the lab frame (i.e., at speed v_e backward relative to the rocket).

At time t, the rocket has mass M and velocity v (in the lab frame). In the next instant $\mathrm{d}t$, it ejects a mass $|\mathrm{d}M| = \dot{m}\,\mathrm{d}t$ at speed v_e relative to the rocket—meaning the ejected gas has lab-frame velocity $v - v_e$.

Applying momentum conservation to the total system (rocket + ejected mass) over this interval:

$$Mv = (M - |\mathrm{d}M|)(v + \mathrm{d}v) + |\mathrm{d}M|(v - v_e). \tag{9.15}$$

Expanding the right side:

$$Mv + M\,\mathrm{d}v - |\mathrm{d}M|\,\mathrm{d}v - |\mathrm{d}M|\,v + |\mathrm{d}M|\,v - |\mathrm{d}M|\,v_e.$$

Canceling Mv and dropping the second-order term $|\mathrm{d}M|\,\mathrm{d}v$:

$$M\,\mathrm{d}v = v_e\,|\mathrm{d}M|. \tag{9.16}$$

Since the rocket's mass decreases, $\mathrm{d}M < 0$ and $|\mathrm{d}M| = -\mathrm{d}M$. Therefore:

$$\mathrm{d}v = -v_e\,\frac{\mathrm{d}M}{M}.$$

Integrating from the initial state $(M_0,\, v_0)$ to the final state $(M_f,\, v_f)$:

Theorem 9.3: Tsiolkovsky Rocket Equation

$$\boxed{\Delta v = v_e \ln\left(\frac{M_0}{M_f}\right).} \tag{9.17}$$

The velocity change depends only on the exhaust speed v_e and the *mass ratio* M_0/M_f, not on the rate at which fuel is burned.

Physical Implications

The logarithmic dependence has profound consequences for space travel:

- To achieve $\Delta v = v_e$: $M_0/M_f = e \approx 2.72$—about 63% of the initial mass must be fuel.
- To achieve $\Delta v = 2v_e$: $M_0/M_f = e^2 \approx 7.4$—about 86% must be fuel.
- To achieve $\Delta v = 3v_e$: $M_0/M_f = e^3 \approx 20$—about 95% must be fuel.

Each additional v_e of velocity requires an *exponentially* larger fraction of the mass to be fuel. This is the fundamental constraint of single-stage rocketry and the reason multi-stage rockets are used: by discarding empty fuel tanks, the "payload" mass M_f is reduced, improving the effective mass ratio.

The Rocket Equation with External Forces

If an external force F_{ext} acts on the rocket (such as gravity), the momentum of the total system is no longer conserved. Equation (9.16) becomes:

$$M\,\mathrm{d}v = v_e\,|\mathrm{d}M| + F_{\text{ext}}\,\mathrm{d}t, \tag{9.18}$$

or equivalently:

$$M\frac{\mathrm{d}v}{\mathrm{d}t} = \underbrace{v_e\,\dot{m}}_{\text{thrust}} + F_{\text{ext}}. \tag{9.19}$$

The quantity $T = v_e\,\dot{m}$ is the **thrust**—the force produced by the rocket engine. For a rocket launched vertically against gravity:

$$M\frac{\mathrm{d}v}{\mathrm{d}t} = v_e\,\dot{m} - Mg.$$

The rocket lifts off only if the thrust exceeds the weight: $v_e\dot{m} > M_0 g$.

Worked Examples

Example 9.16 (Single-stage rocket). A rocket has initial mass $M_0 = 1000\,\text{kg}$, of which 80% is fuel. The exhaust speed is $v_e = 3000\,\text{m/s}$. Find the final speed (ignoring gravity).

Solution. $M_f = 0.20 \times 1000 = 200\,\text{kg}$. The rocket equation gives:

$$\Delta v = v_e \ln(M_0/M_f) = 3000\ln(1000/200) = 3000\ln 5 = 3000(1.609) = 4828\,\text{m/s} \approx 1.61\,v_e.$$

Example 9.17 (Two-stage rocket). Now suppose the same rocket is split into two stages. Stage 1 has $M_0^{(1)} = 1000\,\text{kg}$ with 400 kg of fuel; when it burns out, the 100 kg empty tank is discarded, and Stage 2 (total 500 kg with 400 kg fuel) ignites. Same $v_e = 3000\,\text{m/s}$.

Solution. Stage 1: $M_0^{(1)} = 1000$, $M_f^{(1)} = 600$ (payload + stage 2 + empty tank 1). $\Delta v_1 = 3000\ln(1000/600) = 3000(0.511) = 1532\,\text{m/s}$.

Discard the 100 kg tank. Stage 2: $M_0^{(2)} = 500$, $M_f^{(2)} = 100$. $\Delta v_2 = 3000\ln(500/100) = 3000(1.609) = 4828\,\text{m/s}$.

Total: $\Delta v = 1532 + 4828 = 6360\,\text{m/s} \approx 2.12\,v_e$.

Compare: the single-stage rocket achieved $1.61\,v_e$ with the same fuel and exhaust speed. Staging yields a 32% improvement by discarding dead weight (empty tanks) between stages.

9.7 General Variable-Mass Systems

The rocket equation is a special case of a broader class of **variable-mass problems**: systems that gain or lose mass. The general equation of motion is:

Theorem 9.4: Variable-Mass Equation of Motion

For a body of mass $M(t)$ with velocity v, gaining or losing mass at rate $\dot{m}$, with the accreted/ejected mass having velocity u (in the lab frame):

$$M\frac{\mathrm{d}v}{\mathrm{d}t} = F_{\text{ext}} + (u - v)\frac{\mathrm{d}M}{\mathrm{d}t}. \tag{9.20}$$

The term $(u - v)\,\mathrm{d}M/\mathrm{d}t$ is the **thrust** (for a rocket, $u - v = -v_e$, giving thrust $= +v_e\dot{m}$) or the **drag due to mass accretion** (for a body collecting matter).

Common Mistake 9.2: Variable-Mass Pitfall

A common error is to write "$F = ma$" for a variable-mass system with $m = m(t)$. This is incorrect. The correct approach is to apply momentum conservation to the *entire system* (body + ejected/accreted mass) over an infinitesimal interval, as was done in deriving the rocket equation. The result, Eq. (9.20), contains an extra "thrust" term that $F = m(t)a$ would miss entirely.

Worked Example: The Growing Raindrop

Example 9.18 (Raindrop collecting moisture). A spherical raindrop falls from rest through a cloud, collecting moisture at a rate proportional to its cross-sectional area. If the cloud has uniform water content, show that the raindrop accelerates at $g/7$ at late times.

Solution. Let ρ_w be the water density and ρ_c the cloud water content (mass per unit volume of air). A drop of radius r sweeps out a volume $\pi r^2 v\,\mathrm{d}t$ in time $\mathrm{d}t$, collecting mass $\mathrm{d}m = \rho_c \pi r^2 v\,\mathrm{d}t$:

$$\dot{m} = \rho_c \pi r^2 v.$$

The drop's mass is $m = \frac{4}{3}\pi\rho_w r^3$, so $\dot{m} = 4\pi\rho_w r^2 \dot{r}$, giving:

$$\dot{r} = \frac{\rho_c v}{4\rho_w}.$$

The equation of motion is $\frac{\mathrm{d}}{\mathrm{d}t}(mv) = mg$ (gravity is the only external force; the collected moisture arrives with zero vertical velocity, so there is no thrust term):

$$m\dot{v} + \dot{m}v = mg \quad \Longrightarrow \quad \dot{v} + \frac{\dot{m}}{m}v = g.$$

We seek a self-similar (power-law) solution valid at late times when r_0 is negligible. Assume $v = \alpha t$ and $r = \beta t^2$ (since $\dot{r} \propto v \propto t$, integrating gives $r \propto t^2$). Then $m \propto r^3 \propto t^6$ and $mv \propto t^7$. The momentum equation $\frac{\mathrm{d}}{\mathrm{d}t}(mv) = mg$ gives:

$$7(\text{const})\,t^6 = (\text{const})\,g\,t^6 \quad \Longrightarrow \quad 7\alpha = g \quad \Longrightarrow \quad \boxed{a = \dot{v} = \alpha = \frac{g}{7}.}$$

The drop accelerates at only $g/7$, not g, because six-sevenths of the gravitational impulse goes into accelerating the newly accreted mass from rest to the drop's speed.

9.8 Center of Mass

We have seen that the total momentum of a system equals $\boldsymbol{P} = M\boldsymbol{v}_{\text{cm}}$, where $\boldsymbol{v}_{\text{cm}}$ is the velocity of the center of mass. But where, exactly, *is* the center of mass? In this section we define it precisely for both discrete and continuous mass distributions and develop its most important dynamical properties. The *center-of-mass reference frame*—a powerful tool for simplifying collision problems—will be developed in Chapter 10.

Definition for Discrete Systems

Definition 9.3: Center of Mass

For a system of N particles with masses $m_1, m_2, \ldots, m_N$ at positions $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N$:

$$\boxed{\boldsymbol{R}_{\text{cm}} = \frac{1}{M}\sum_{i=1}^{N} m_i\,\boldsymbol{r}_i, \qquad M = \sum_{i=1}^{N} m_i.} \tag{9.21}$$

In component form: $x_{\text{cm}} = \frac{1}{M}\sum m_i x_i$, $y_{\text{cm}} = \frac{1}{M}\sum m_i y_i$, $z_{\text{cm}} = \frac{1}{M}\sum m_i z_i$.

The center of mass is the *mass-weighted average position* of all particles. It lies closer to heavier particles and, for a symmetric distribution, at the geometric center.

Example 9.19 (Three particles). Masses $m_1 = 2\,\text{kg}$ at $(0,0)$, $m_2 = 3\,\text{kg}$ at $(4,0)$ m, $m_3 = 5\,\text{kg}$ at $(2,3)$ m. Total mass $M = 10$ kg.

$$x_{\text{cm}} = \frac{2(0)+3(4)+5(2)}{10} = \frac{22}{10} = 2.2\,\text{m}, \qquad y_{\text{cm}} = \frac{2(0)+3(0)+5(3)}{10} = 1.5\,\text{m}.$$

The CM is at $(2.2, 1.5)$—closer to the heaviest mass (m_3), as expected.

Definition for Continuous Distributions

For a continuous body, the sums become integrals:

$$\boldsymbol{R}_{\text{cm}} = \frac{1}{M}\int \boldsymbol{r}\,\mathrm{d}m, \qquad M = \int \mathrm{d}m, \tag{9.22}$$

where $\mathrm{d}m = \rho\,\mathrm{d}V$ (volume density), $\mathrm{d}m = \sigma\,\mathrm{d}A$ (surface density), or $\mathrm{d}m = \lambda\,\mathrm{d}\ell$ (linear density), depending on the geometry.

Strategy 9.2: Finding the CM of a Continuous Body

1. Choose coordinates that exploit symmetry. If the body has an axis of symmetry, the CM lies on that axis—skip the vanishing integrals.
2. Express $\mathrm{d}m$ in terms of a single integration variable when possible.
3. Compute $M = \int \mathrm{d}m$.
4. Compute each coordinate: $x_{\text{cm}} = \frac{1}{M}\int x\,\mathrm{d}m$, etc.

Example 9.20 (Non-uniform rod). A rod of length L has linear mass density $\lambda(x) = \lambda_0(1 + x/L)$. Find M and x_{cm}.

Solution. $M = \int_0^L \lambda_0(1 + x/L)\,\mathrm{d}x = \lambda_0\big[x + x^2/(2L)\big]_0^L = \frac{3}{2}\lambda_0 L$.

$x_{\rm cm} = \frac{1}{M}\int_0^L x\,\lambda_0(1+x/L)\,\mathrm{d}x = \frac{\lambda_0}{M}\big[\frac{x^2}{2}+\frac{x^3}{3L}\big]_0^L = \frac{\lambda_0}{M}\cdot\frac{5L^2}{6} = \frac{5L}{9}$.

Since the rod is denser at the right end, $x_{\rm cm} > L/2$, as expected. In the uniform limit ($\lambda = \text{const}$), we recover $x_{\rm cm} = L/2$.

Example 9.21 (Semicircular wire). Find the CM of a uniform semicircular wire of radius R.

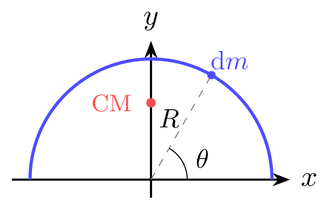


Figure 9.8.1: Center of mass of a uniform semicircular wire of radius R.

Solution. By symmetry, $x_{\rm cm} = 0$. Parameterize by $\theta \in [0,\pi]$: $y = R\sin\theta$, $\mathrm{d}m = \lambda R\,\mathrm{d}\theta$, $M = \lambda\pi R$.

$$y_{\rm cm} = \frac{1}{M}\int_0^\pi (R\sin\theta)\,\lambda R\,\mathrm{d}\theta = \frac{\lambda R^2}{\lambda\pi R}\int_0^\pi \sin\theta\,\mathrm{d}\theta = \frac{R}{\pi}(2) = \frac{2R}{\pi} \approx 0.637R.$$

The Negative-Mass Trick

For bodies with holes or missing pieces, computing the CM directly is tedious. A far more elegant method treats the hole as having *negative mass*. If the full body (without hole) has mass M and CM at $\boldsymbol{R}$, and the removed piece has mass m_h and CM at $\boldsymbol{r}_h$, then the remaining piece (mass $M - m_h$) has CM at:

$$\boldsymbol{R}_{\rm cm} = \frac{M\boldsymbol{R} - m_h\boldsymbol{r}_h}{M - m_h}. \tag{9.23}$$

Example 9.22 (Disk with a hole). A uniform disk of radius R and mass M has a circular hole of radius $R/2$ cut from it, centered at distance $R/2$ from the center. Find the CM of the remaining piece.

Solution. Full disk: CM at origin, mass M. Hole: CM at $(R/2, 0)$, mass $m_h = M(R/2)^2/R^2 = M/4$. Remaining piece: mass $3M/4$.

$$x_{\rm cm} = \frac{M(0) - (M/4)(R/2)}{3M/4} = \frac{-MR/8}{3M/4} = -\frac{R}{6}.$$

The CM shifts *away* from the hole, toward the heavier side.

Motion of the Center of Mass

Differentiating the definition $\boldsymbol{R}_{\rm cm} = \frac{1}{M}\sum m_i\boldsymbol{r}_i$ with respect to time:

$$\boldsymbol{v}_{\rm cm} = \frac{1}{M}\sum m_i\boldsymbol{v}_i = \frac{\boldsymbol{P}}{M}. \tag{9.24}$$

This confirms $\boldsymbol{P} = M\boldsymbol{v}_{\rm cm}$. Differentiating once more:

$$\boldsymbol{a}_{\rm cm} = \frac{1}{M}\sum m_i\boldsymbol{a}_i = \frac{1}{M}\sum \boldsymbol{F}_i. \tag{9.25}$$

Internal forces cancel in pairs (Newton's third law), leaving only external forces:

Theorem 9.5: Newton's Second Law for the Center of Mass

$$\boxed{\boldsymbol{F}_{\text{ext}} = M\boldsymbol{a}_{\text{cm}}.} \tag{9.26}$$

The center of mass of a system moves as if all the mass were concentrated there and all external forces acted on that point.

This has powerful consequences:

Key Point 9.5: CM Motion Is Independent of Internal Forces

No internal interaction (collision, explosion, spring force, tension) can change the motion of the center of mass. If a firework follows a parabolic trajectory under gravity, then after it explodes, the *center of mass of the fragments* continues along the same parabola. The individual pieces fly in all directions, but their mass-weighted average position traces the original path.

Problems

Problem 9.1 ⋆

A 0.145 kg baseball is pitched at 40 m/s and hit directly back at 50 m/s. Find the impulse delivered by the bat. If the contact time is 1.0 ms, find the average force.

Problem 9.2 ⋆

A 70 kg astronaut floating at rest in space throws a 5.0 kg tool kit at 6.0 m/s. (a) Find the astronaut's recoil speed. (b) Find the kinetic energy of the system after the throw. Where did this energy come from?

Problem 9.3 ⋆⋆

A 0.40 kg ball traveling at 12 m/s strikes a wall and bounces back at 9.0 m/s. Contact time is 0.020 s. (a) Find the impulse. (b) Find the average force. (c) Compare the impulse to that of an identical ball that sticks to the wall.

Problem 9.4 ⋆⋆

A 60 kg person jumps from a height of 2.0 m. (a) Find the speed just before landing. (b) If they land stiff-legged and stop in 0.01 s, find the average force on their legs. (c) If they bend their knees and stop in 0.5 s, find the average force. Comment on the importance of bending your knees.

Problem 9.5 ⋆⋆⋆

A ball of mass m launched at speed V and angle θ lands on a stationary sledge of mass M on a frozen lake, sticking to it. (a) Find V_f of the sledge. (b) Find the kinetic energy converted to other forms.

Problem 9.6 ⋆⋆⋆

A force $F(t) = F_0 e^{-t/\tau}$ acts on a mass m initially at rest. (a) Find the total impulse from $t = 0$ to $t = \infty$. (b) Find the final speed. (c) Find $v(t)$. (d) Find the total work done by the force and verify it equals $\frac{1}{2}mv_\infty^2$.

Problem 9.7 ⋆⋆⋆

A 10 kg object at rest explodes into three pieces. A 3 kg piece moves north at 8 m/s; a 4 kg piece moves east at 6 m/s. Find the speed and direction of the third piece.

Problem 9.8 ⋆⋆⋆

(*Ballistic pendulum.*) A 0.010 kg bullet traveling at 400 m/s embeds in a 2.0 kg wooden block suspended as a pendulum. (a) Find the speed of the block+bullet immediately after impact. (b) Find the maximum height of the swing. (c) What fraction of the bullet's kinetic energy is lost in the collision?

Problem 9.9 ⋆⋆⋆

A water hose delivers water at rate $\dot{m} = 2.0$ kg/s at speed $v = 20$ m/s against a wall. (a) If the water does not bounce back, find the force on the wall. (b) If the water bounces back elastically, find the force. (c) If the water is deflected at 90° (sideways), find the force on the wall.

Problem 9.10 ⋆⋆⋆

An open railroad car of mass M rolls at speed v_0 under a hopper that drops sand vertically at rate $\dot{m}$. (a) Find $v(t)$. (b) Find the kinetic energy as a function of time and show it decreases. (c) At what rate is energy dissipated? Where does it go?

Problem 9.11 ⋆⋆⋆⋆

Derive the Tsiolkovsky rocket equation. A rocket of initial mass M_0 ejects mass at exhaust speed v_e. (a) Show that $\Delta v = v_e \ln(M_0/M_f)$. (b) If 80% of the initial mass is fuel, find the final speed in units of v_e. (c) For a rocket launched vertically against gravity with constant $\dot{m}$, write the equation of motion and show that liftoff requires $v_e \dot{m} > M_0 g$.

Problem 9.12 ⋆⋆⋆⋆

A uniform chain of mass m and length L is held vertically with its lower end just touching a scale. It is released. (a) Show that the falling portion has speed $v = \sqrt{2gy}$ when a length y has piled up. (b) Show the total force on the scale is $F = 3mgy/L$. (c) Find F at the instant the last link hits the scale and compare with the static weight mg.

Problem 9.13 ⋆⋆⋆⋆

A two-stage rocket has total initial mass M_0. Each stage has the same fuel fraction f (fraction of that stage's initial mass that is fuel) and the same exhaust speed v_e. The empty first-stage casing has mass m_c and is discarded after burnout. (a) Show that the total Δv is $\Delta v = -2v_e \ln(1-f)$ if the casing mass is negligible. (b) Explain qualitatively why staging is more efficient than a single stage with the same total fuel.

Problem 9.14 ⋆⋆⋆⋆⋆

A raindrop falls through a cloud, collecting moisture at a rate proportional to its cross-sectional area. The drop starts from rest with radius $r_0 \to 0$ and the cloud has uniform water content. (a) Show that $\dot{r} = \rho_c v/(4\rho_w)$. (b) By seeking a self-similar solution $v \propto t$, $r \propto t^2$, show that the acceleration approaches $g/7$ at late times. (c) What fraction of the gravitational force goes into accelerating the drop versus accelerating the accreted mass?

Problem 9.15 ⋆⋆⋆

A uniform disk of radius R has a circular hole of radius $R/3$ cut from it, with the hole centered at distance $R/3$ from the disk's center along the x-axis. (a) Find the center of mass of the remaining piece using the negative-mass method. (b) If this piece is placed on a frictionless surface and given an impulse $J\,\hat{\imath}$ at its center of mass, find its subsequent velocity.

Chapter 10

Collisions

A **collision** is a short-duration interaction in which two or more objects exert large forces on each other. During the brief collision time, these internal impulsive forces vastly exceed any external forces (gravity, friction), so momentum is conserved to excellent approximation. Whether kinetic energy is also conserved depends on the *type* of collision.

In this chapter we develop the physics of elastic, inelastic, and perfectly inelastic collisions in one and two dimensions. We then introduce the **coefficient of restitution**, which characterizes real-world collisions, and the **center-of-mass reference frame**, which reduces elastic collisions to a trivial velocity reversal. The center-of-mass definition and its basic properties were developed in Section 9.8; here we exploit the CM frame as a computational tool.

10.1 Types of Collisions

Definition 10.1: Collision Classification

Elastic: Both momentum and kinetic energy are conserved. No energy is converted to heat, sound, or deformation. Atomic and subatomic collisions are often elastic; macroscopic ones rarely are (hardened steel balls and billiard balls come close).
Inelastic: Momentum is conserved but kinetic energy is not. Some kinetic energy is converted to other forms (heat, sound, deformation, internal excitation). Most real-world collisions are inelastic.
Perfectly inelastic: The objects stick together after the collision. This gives the *maximum* kinetic energy loss consistent with momentum conservation. The objects move with a common final velocity.

Key Point 10.1: Momentum Is Always Conserved in Collisions

Regardless of the collision type (elastic, inelastic, or perfectly inelastic) momentum is conserved whenever external impulses are negligible during the collision. **Kinetic energy** is conserved *only* in elastic collisions. In **explosions** (the reverse of perfectly inelastic collisions), kinetic energy *increases* as internal energy is converted to kinetic energy.

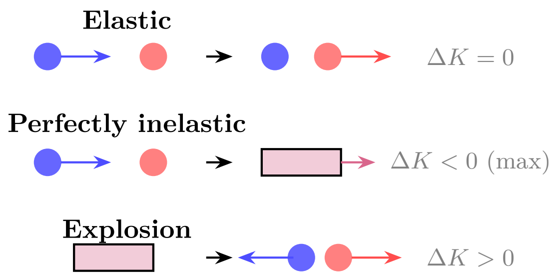


Figure 10.1.1: Three collision types. Elastic: KE conserved. Perfectly inelastic: objects stick, max KE loss. Explosion: reverse of perfectly inelastic, KE increases.

10.2 Elastic Collisions in One Dimension

Derivation of the Elastic Collision Formulas

Consider a projectile of mass m_1 with velocity v_0 striking a stationary target of mass m_2. Two conservation laws apply:

Momentum: $m_1 v_0 = m_1 v_1' + m_2 v_2'$.
Kinetic energy: $\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$.

Solving two equations in two unknowns is straightforward but tedious. The key trick is to *factor* the energy equation. Rearranging:

$$m_1(v_0 - v_1') = m_2 v_2' \qquad \text{(momentum)},$$

$$m_1(v_0^2 - v_1'^2) = m_2 v_2'^2 \qquad \text{(energy)}.$$

The left side of the energy equation factors: $m_1(v_0 - v_1')(v_0 + v_1') = m_2 v_2'^2$. Dividing the energy equation by the momentum equation (valid since $v_0 \neq v_1'$ in a real collision):

$$\boxed{v_0 + v_1' = v_2'.} \tag{10.1}$$

This is the **relative velocity reversal condition**: the relative speed of approach equals the relative speed of separation. In the general case (both objects initially moving): $v_1' - v_2' = -(v_1 - v_2)$. This single *linear* equation replaces the *quadratic* energy equation, making the system trivially solvable.

Substituting $v_2' = v_0 + v_1'$ into the momentum equation:

$$m_1 v_0 = m_1 v_1' + m_2(v_0 + v_1') \implies (m_1 + m_2) v_1' = (m_1 - m_2) v_0.$$

Theorem 10.1: 1D Elastic Collision (Target at Rest)

$$\boxed{v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_0, \qquad v_2' = \frac{2m_1}{m_1 + m_2} v_0.} \tag{10.2}$$

Key Point 10.2: Special Cases of 1D Elastic Collisions

Equal masses ($m_1 = m_2$): $v_1' = 0$, $v_2' = v_0$. Complete energy transfer: the projectile stops dead. This is why Newton's cradle works.
Heavy projectile ($m_1 \gg m_2$): $v_1' \approx v_0$, $v_2' \approx 2v_0$. The target is launched at roughly twice

the projectile's speed, while the projectile barely slows.
Light projectile ($m_1 \ll m_2$): $v_1' \approx -v_0$, $v_2' \approx 0$. The projectile bounces back at nearly its original speed; the heavy target barely moves. (Think: tennis ball bouncing off a wall.)

Energy Transfer

The fraction of kinetic energy transferred to the target is:

$$\frac{K_2'}{K_0} = \frac{4m_1m_2}{(m_1+m_2)^2} = \frac{4r}{(1+r)^2}, \qquad r \equiv m_2/m_1. \tag{10.3}$$

This function is maximized (= 1, complete transfer) when $r = 1$ ($m_1 = m_2$) and drops to zero in either extreme mass ratio ($r \to 0$ or $r \to \infty$).

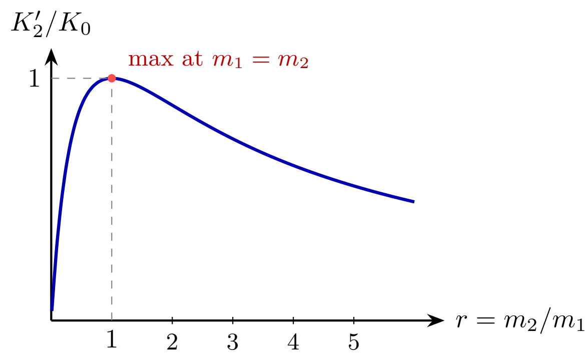


Figure 10.2.1: Fraction of kinetic energy transferred to the target in a 1D elastic collision, as a function of the mass ratio $r = m_2/m_1$. Maximum transfer occurs at $r = 1$ (equal masses).

This result has practical importance in **neutron moderation** in nuclear reactors. Fast neutrons from fission must be slowed ("moderated") to increase the probability of further fission. Each elastic collision transfers a fraction $4r/(1+r)^2$ of the neutron's energy to the moderator nucleus. For hydrogen ($r = 1$): 100% transfer per head-on collision. For carbon-12 ($r = 12$): $4(12)/169 \approx 28\%$ per collision. Water (hydrogen-rich) is a much more efficient moderator than graphite (carbon), though graphite has other engineering advantages.

General Case: Both Objects Moving

When both objects are initially moving, the formulas generalize to:

$$v_1' = \frac{m_1-m_2}{m_1+m_2}v_1 + \frac{2m_2}{m_1+m_2}v_2, \qquad v_2' = \frac{2m_1}{m_1+m_2}v_1 + \frac{m_2-m_1}{m_1+m_2}v_2. \tag{10.4}$$

These are most easily derived by transforming to the CM frame (Section 10.6), reversing velocities, and transforming back.

Worked Examples

Example 10.1 (Superball bounce). A small ball of mass m sits atop a large ball of mass $M \gg m$, and both are dropped from height h. After the large ball bounces elastically off the floor, it collides elastically with the small ball. Show that the small ball can reach a height of $9h$.

Solution. Both balls arrive at the floor with speed $v_0 = \sqrt{2gh}$. The large ball bounces off the floor first and moves upward at v_0. An instant later, it collides with the small ball, which is still moving downward at v_0.

In the frame of the large ball (approximately the CM frame for $M \gg m$), the small ball approaches at $2v_0$. After an elastic collision with a much heavier object, the small ball reverses: it leaves at $2v_0$ (upward) in the large ball's frame. Transforming back to the lab frame:

$$v'_{\text{small}} = 2v_0 + v_0 = 3v_0.$$

The maximum height is:

$$h' = \frac{(3v_0)^2}{2g} = \frac{9v_0^2}{2g} = 9h.$$

The small ball reaches 9 times the original drop height! This dramatic energy transfer is the principle behind "astro-blaster" toys.

Example 10.2 (Finding a mass from collision data). In a 1D elastic collision, puck 1 (mass m_1) at speed V hits stationary puck 2 (mass m_2). Puck 2 leaves at $V/2$. Find m_2 in terms of m_1 and find v'_1.

Solution. From $v'_2 = \frac{2m_1}{m_1+m_2}V = V/2$: $4m_1 = m_1 + m_2$, so $m_2 = 3m_1$. The relative velocity reversal gives $v'_1 = v'_2 - V = V/2 - V = -V/2$. Verify: $v'_1 = \frac{m_1-3m_1}{4m_1}V = -V/2$. ✓

10.3 Perfectly Inelastic Collisions

When two objects stick together, momentum conservation alone determines the outcome (one equation, one unknown):

$$\boxed{v_f = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.} \tag{10.5}$$

Energy Loss

The kinetic energy lost is:

$$|\Delta K| = K_i - K_f = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(v_1 - v_2)^2 = \frac{1}{2}\mu\, v_{\text{rel}}^2, \tag{10.6}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the **reduced mass** of the two-body system. This elegant result states that the energy lost depends only on the reduced mass and the relative speed of approach, not on the individual masses or speeds separately.

Key Point 10.3: The Reduced Mass

The reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ (equivalently, $1/\mu = 1/m_1 + 1/m_2$) arises naturally whenever a two-body problem is reduced to an equivalent one-body problem. Note that $\mu < \min(m_1, m_2)$ always, and $\mu \to m_1$ when $m_2 \to \infty$. The reduced mass will appear again in orbital mechanics and oscillating systems.

Proof of Eq. (10.6). $K_i = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$. $K_f = \frac{1}{2}(m_1 + m_2)v_f^2 = \frac{(m_1 v_1 + m_2 v_2)^2}{2(m_1+m_2)}$.

$$\Delta K = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} - \frac{(m_1 v_1 + m_2 v_2)^2}{2(m_1 + m_2)}$$

$$= \frac{(m_1 + m_2)(m_1 v_1^2 + m_2 v_2^2) - (m_1 v_1 + m_2 v_2)^2}{2(m_1 + m_2)}$$
$$= \frac{m_1 m_2 (v_1^2 - 2v_1 v_2 + v_2^2)}{2(m_1 + m_2)} = \frac{1}{2}\mu\,(v_1 - v_2)^2. \qquad \square$$

Worked Examples

Example 10.3 (Ballistic pendulum). A bullet of mass m at speed v_0 embeds in a block of mass M on a string of length ℓ. Find (a) the velocity after impact, (b) the maximum height h, and (c) the fraction of KE lost.

Solution. (a) Perfectly inelastic collision: $v_f = mv_0/(m + M)$.

(b) After the collision, energy conservation: $\frac{1}{2}(m + M)v_f^2 = (m + M)gh$, so $h = v_f^2/(2g) = m^2 v_0^2/[2g(m + M)^2]$.

(c) $|\Delta K|/K_i = 1 - K_f/K_i = 1 - \frac{(m+M)v_f^2}{mv_0^2} = 1 - \frac{m}{m+M} = \frac{M}{m+M}$.

For a typical bullet ($m = 10\,\mathrm{g}$) and block ($M = 2\,\mathrm{kg}$): the fraction lost is $2000/2010 = 99.5\%$. The ballistic pendulum "wastes" nearly all the kinetic energy as heat and deformation, but this is precisely what makes it work: the measured height h is sensitive to the tiny surviving KE, from which v_0 can be extracted.

Example 10.4 (Maximum KE loss). Show that a perfectly inelastic collision gives the maximum possible KE loss consistent with momentum conservation.

Solution. After any collision, the total momentum is $p = m_1 v_1' + m_2 v_2'$. The final KE is $K_f = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$. By König's theorem:

$$K_f = \frac{p^2}{2(m_1 + m_2)} + K_{\text{int}}, \qquad K_{\text{int}} = \frac{1}{2}\mu\, v_{\text{rel}}'^2.$$

The first term is fixed by momentum conservation (it equals the CM kinetic energy). The second term, K_{int}, is the "internal" KE in the CM frame and is minimized when $v_{\text{rel}}' = 0$, i.e., when the objects move together. That is the definition of a perfectly inelastic collision.

10.4 Two-Dimensional Collisions

In two dimensions, momentum conservation provides *two* component equations (x and y). For an elastic collision, energy conservation provides a third. With four unknowns (two final velocity vectors, each with magnitude and direction), one additional constraint is needed—typically a scattering angle is measured or specified.

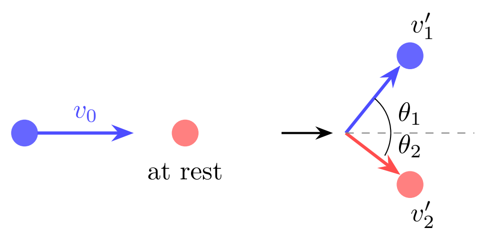


Figure 10.4.1: A 2D collision. The projectile (blue) scatters at angle θ_1; the target (red) recoils at angle θ_2.

Equal-Mass Elastic Collisions: The 90° Rule

Theorem 10.2: Equal-Mass Elastic 2D Collision

When two equal-mass particles undergo an elastic collision with one initially at rest, the final velocities are **perpendicular**: $\boldsymbol{v}_1' \cdot \boldsymbol{v}_2' = 0$.

Proof. From momentum conservation, $\boldsymbol{v}_0 = \boldsymbol{v}_1' + \boldsymbol{v}_2'$ (masses cancel). Squaring: $v_0^2 = v_1'^2 + 2\boldsymbol{v}_1' \cdot \boldsymbol{v}_2' + v_2'^2$. Energy conservation (masses cancel): $v_0^2 = v_1'^2 + v_2'^2$. Subtracting: $2\boldsymbol{v}_1' \cdot \boldsymbol{v}_2' = 0$, so $\boldsymbol{v}_1' \perp \boldsymbol{v}_2'$.

This is beautifully visible in billiards: after a non-head-on collision between two balls of equal mass, the cue ball and the object ball always scatter at a right angle (in the absence of spin).

Example 10.5 (Billiards at 30°). Ball 1 (speed v_0) hits stationary ball 2 (equal mass). Ball 2 deflects at $\theta_2 = 30°$. Find both final speeds and ball 1's angle.

Solution. By the 90° rule: $\theta_1 = 60°$. Momentum (x): $v_0 = v_1' \cos 60° + v_2' \cos 30° = \frac{1}{2}v_1' + \frac{\sqrt{3}}{2}v_2'$. Momentum ($y$): $0 = v_1' \sin 60° - v_2' \sin 30° = \frac{\sqrt{3}}{2}v_1' - \frac{1}{2}v_2'$, giving $v_2' = \sqrt{3}\,v_1'$.

Substituting: $v_0 = \frac{1}{2}v_1' + \frac{3}{2}v_1' = 2v_1'$, so $v_1' = v_0/2$ and $v_2' = \sqrt{3}\,v_0/2$.

Check energy: $v_1'^2 + v_2'^2 = v_0^2/4 + 3v_0^2/4 = v_0^2$. ✓

10.5 The Coefficient of Restitution

Real collisions are neither perfectly elastic nor perfectly inelastic. The **coefficient of restitution** e parameterizes how "bouncy" a collision is:

Definition 10.2: Coefficient of Restitution

For a 1D collision:

$$\boxed{e = -\frac{v_1' - v_2'}{v_1 - v_2} = \frac{\text{relative speed of separation}}{\text{relative speed of approach}}.} \tag{10.7}$$

Elastic: $e = 1$. Perfectly inelastic: $e = 0$. Real collisions: $0 < e < 1$.

When e is given, the pair of equations (momentum conservation and the restitution condition) are *both linear* and easily solved:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2', \tag{10.8}$$

$$v_1' - v_2' = -e(v_1 - v_2). \tag{10.9}$$

This is far simpler than using the (quadratic) energy equation directly.

Energy Loss in Terms of e

The kinetic energy lost in a collision with restitution coefficient e is:

$$|\Delta K| = \frac{1}{2}\mu\, v_{\text{rel}}^2 (1 - e^2). \tag{10.10}$$

For $e = 1$: $|\Delta K| = 0$ (elastic). For $e = 0$: $|\Delta K| = \frac{1}{2}\mu v_{\text{rel}}^2$ (maximum loss).

Example 10.6 (Bouncing ball with restitution). A ball of mass m is dropped from height h onto a stationary block of mass M on a frictionless floor. The collision has coefficient of restitution e. Find velocities after the collision and the bounce height.

Solution. The ball hits at $v_b = \sqrt{2gh}$ (downward); the block is initially at rest. Using Eqs. (10.8)–(10.9):

$$v_1' = \frac{m - eM}{m + M}\sqrt{2gh}, \qquad v_2' = \frac{m(1+e)}{m+M}\sqrt{2gh}.$$

The ball bounces to height $h' = v_1'^2/(2g) = h\left(\frac{m-eM}{m+M}\right)^2$.

For $m = M$, $e = 0.8$: $v_1' = \frac{1-0.8}{2}\sqrt{2gh} = 0.1\sqrt{2gh}$, so $h' = 0.01h$—only 1% of the original height.

10.6 The Center-of-Mass Reference Frame

The **center-of-mass (CM) frame** is the reference frame in which the total momentum of the system is zero. It is the natural frame for analyzing collisions because the physics simplifies dramatically: in the CM frame, elastic collisions reduce to a trivial velocity reversal.

Definition 10.3: Center of Mass Frame

The CM frame moves with velocity $\boldsymbol{v}_{\text{cm}} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1+m_2}$ relative to the lab frame. Velocities in the CM frame are:

$$\boldsymbol{v}_i^{\text{cm}} = \boldsymbol{v}_i - \boldsymbol{v}_{\text{cm}}. \tag{10.11}$$

By construction, the total momentum vanishes: $\sum m_i \boldsymbol{v}_i^{\text{cm}} = \boldsymbol{0}$.
For two bodies: $m_1\boldsymbol{v}_1^{\text{cm}} = -m_2\boldsymbol{v}_2^{\text{cm}}$ (the momenta are equal and opposite).

Elastic Collisions in the CM Frame

In the CM frame, the total momentum is zero both before and after the collision. Energy conservation then requires that the *speeds* are unchanged, only the directions reverse. For a 1D elastic collision:

$$v_1'^{\text{cm}} = -v_1^{\text{cm}}, \qquad v_2'^{\text{cm}} = -v_2^{\text{cm}}. \tag{10.12}$$

Each particle simply bounces back at the same speed. This is the fundamental reason why the CM frame is so powerful: the complicated elastic collision formulas in the lab frame become a trivial sign flip in the CM frame.

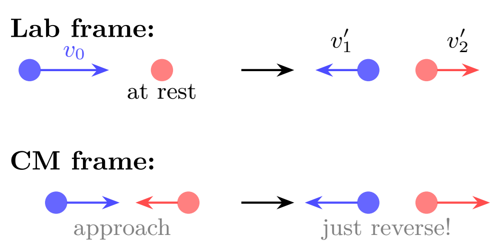


Figure 10.6.1: In the CM frame, an elastic collision simply reverses each velocity. Transforming back to the lab frame recovers the full elastic collision formulas.

Example 10.7 (Elastic collision via the CM frame). Mass m_1 at v_0 hits stationary m_2. CM velocity: $v_{\text{cm}} = m_1 v_0/(m_1+m_2)$.

In the CM frame: $v_1^{\text{cm}} = v_0 - v_{\text{cm}} = \frac{m_2 v_0}{m_1+m_2}$, $\quad v_2^{\text{cm}} = -v_{\text{cm}} = \frac{-m_1 v_0}{m_1+m_2}$.

After elastic collision (reverse): $v_1'^{\text{cm}} = \frac{-m_2 v_0}{m_1+m_2}$, $\quad v_2'^{\text{cm}} = \frac{m_1 v_0}{m_1+m_2}$.

Back to lab: $v_1' = v_1'^{\text{cm}} + v_{\text{cm}} = \frac{m_1-m_2}{m_1+m_2}\,v_0$, $\quad v_2' = v_2'^{\text{cm}} + v_{\text{cm}} = \frac{2m_1}{m_1+m_2}\,v_0$.

This reproduces Theorem 10.1 without solving simultaneous equations.

König's Theorem

Theorem 10.3: König's Theorem

The total kinetic energy of a system decomposes as:

$$\boxed{K_{\text{total}} = \tfrac{1}{2}Mv_{\text{cm}}^2 + K_{\text{int}}, \qquad K_{\text{int}} = \sum_i \tfrac{1}{2}m_i|\boldsymbol{v}_i^{\text{cm}}|^2.} \tag{10.13}$$

For a two-body system: $K_{\text{int}} = \frac{1}{2}\mu\, v_{\text{rel}}^2$, where μ is the reduced mass and $v_{\text{rel}} = |v_1 - v_2|$.

Proof. Write $\boldsymbol{v}_i = \boldsymbol{v}_{\text{cm}} + \boldsymbol{v}_i^{\text{cm}}$ and expand:

$$K = \sum_i \tfrac{1}{2}m_i|\boldsymbol{v}_{\text{cm}} + \boldsymbol{v}_i^{\text{cm}}|^2 = \tfrac{1}{2}Mv_{\text{cm}}^2 + \boldsymbol{v}_{\text{cm}} \cdot \underbrace{\textstyle\sum_i m_i \boldsymbol{v}_i^{\text{cm}}}_{=\boldsymbol{0}} + \sum_i \tfrac{1}{2}m_i|\boldsymbol{v}_i^{\text{cm}}|^2.$$

The cross term vanishes because the total momentum in the CM frame is zero.

König's theorem gives a clean interpretation of energy loss in collisions:

- The CM kinetic energy $\frac{1}{2}Mv_{\text{cm}}^2$ is *never* changed by a collision (it is determined by the conserved total momentum).
- Only the internal kinetic energy K_{int} can change.
- In a perfectly inelastic collision, $K_{\text{int}} \to 0$ (no relative motion), so the energy lost is exactly $\frac{1}{2}\mu v_{\text{rel}}^2$.
- In an elastic collision, K_{int} is unchanged (relative speed is preserved).

10.7 Collision Problem-Solving Summary

Strategy 10.1: Solving Collision Problems

1. **Identify the type**: elastic ($e = 1$), perfectly inelastic ($e = 0$, objects stick), or general inelastic ($0 < e < 1$)?
2. **Draw before/after diagrams** and define a positive direction.
3. **Write momentum conservation**: $m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$ (component by component in 2D).
4. **Write a second equation**:
   - Elastic: $v_1' - v_2' = -(v_1 - v_2)$ (relative velocity reversal).
   - Perfectly inelastic: $v_1' = v_2'$ (stick together).
   - General: $v_1' - v_2' = -e(v_1 - v_2)$ (coefficient of restitution).
5. **Solve** the resulting system of linear equations.
6. **Check**: does KE decrease (inelastic) or stay the same (elastic)? Does the CM velocity remain unchanged?

Problems

Problem 10.1 ⋆⋆

In a 1D elastic collision, puck 1 (m_1) at speed V hits stationary puck 2 (m_2). Find m_2 (in terms of m_1) such that puck 2 leaves with speed $V/2$. Find v_1'.

Problem 10.2 ⋆⋆⋆

(*Ballistic pendulum.*) A bullet of mass m at speed v_0 embeds in a block of mass M on a string of length ℓ. (a) Find the velocity immediately after collision. (b) Find the maximum height h. (c) Derive v_0 from measured h. (d) What fraction of KE is lost?

Problem 10.3 ⋆⋆⋆

Two equal-mass billiard balls undergo an elastic collision. Ball 1 at v_0 hits stationary ball 2, which deflects at 30° from the original direction. (a) Find ball 1's deflection angle. (b) Find both speeds after the collision.

Problem 10.4 ⋆⋆⋆

A photon of frequency ν bounces back and forth between two mirrors separated by L. Photon momentum is $p = h\nu/c$. (a) Find the average force on one mirror. (b) Find the work done by the photon as the mirrors move apart from L to $2L$.

Problem 10.5 ⋆⋆⋆⋆

1D elastic collision: m_1 at v_0, m_2 at rest. (a) Derive the fraction of KE transferred as $f(r)$ where $r = m_2/m_1$. (b) Show f is maximized at $r = 1$. (c) A neutron (m_n) collides head-on with a carbon-12 nucleus ($12m_n$): what fraction of the neutron's energy is transferred per collision? How many collisions are needed to reduce the neutron's energy to $1/e$ of its original value?

Problem 10.6 ⋆⋆⋆

A ball of mass m is dropped from height h onto a stationary block of mass M on a frictionless floor. The collision has coefficient of restitution e. (a) Find the velocity of each object immediately after the collision. (b) Find the height to which the ball bounces. (c) Evaluate for $m = M$ and $e = 0.8$.

Problem 10.7 ⋆⋆⋆⋆

A cannon of mass M on a frictionless surface fires a cannonball of mass m at speed v_0 relative to the cannon at angle θ above horizontal. (a) Find the recoil speed of the cannon. (b) Find the range of the cannonball (as measured by a ground observer). (c) Show that the range is maximized at $\theta = 45°$ and find the ratio of this maximum range to the range of a cannonball fired at the same relative speed from a fixed cannon.

Problem 10.8 ⋆⋆⋆

A 2.0 kg block moving at 5.0 m/s on a frictionless surface collides with and sticks to a 3.0 kg block at rest. (a) Find the final velocity. (b) Find the kinetic energy lost. (c) Express the energy lost using the reduced mass and verify it equals $\frac{1}{2}\mu v_{\text{rel}}^2$.

Problem 10.9 ⋆⋆⋆⋆

(*Superball bounce.*) A small ball of mass m sits atop a large ball of mass $M \gg m$. Both are dropped from height h. The large ball bounces elastically off the floor, then immediately collides elastically with the small ball. (a) Show the small ball reaches height $9h$. (b) What happens if M is only twice m? Find the exact height.

Problem 10.10 ⋆⋆⋆⋆

A firecracker of mass $3m$ is launched vertically at speed v_0. At maximum height, it explodes into three equal pieces: one has zero velocity, a second moves horizontally at $2v_0$. (a) Find the velocity of the third piece. (b) Find the landing positions of all three pieces relative to the launch point. (c) Verify that the CM of the landing positions equals the landing position of the unexploded firecracker.

Problem 10.11 ⋆⋆⋆⋆

Prove König's theorem for a system of N particles. That is, show that $K = \frac{1}{2}Mv_{\text{cm}}^2 + \sum_i \frac{1}{2}m_i|\boldsymbol{v}_i^{\text{cm}}|^2$ where the cross term vanishes because total CM-frame momentum is zero. Then specialize to two particles and show $K_{\text{int}} = \frac{1}{2}\mu v_{\text{rel}}^2$.

Problem 10.12 ⋆⋆⋆⋆⋆

(Newton's cradle analysis.) In Newton's cradle, N identical balls of mass m hang in a row touching each other. Ball 1 is pulled back and released at speed v_0. (a) Explain why only ball N flies off (not, say, balls $N-1$ and N at half speed each). (b) If two balls are pulled back and released at v_0, show that two balls fly off the other side at v_0. (c) Why do all intermediate balls remain essentially stationary?

Chapter 11

Rotation of Rigid Bodies

Up to now we have treated every object as a **point particle**—a body whose size and shape are irrelevant. That idealization works perfectly for translational motion, but it fails the moment an object spins, rolls, or tumbles. A spinning figure skater, a rolling wheel, and a wobbling top all demand a theory of **rotational motion**.

The mathematical structure of rotation is a beautiful parallel to translation: every translational quantity has a rotational analogue. This chapter develops the **kinematics** of rotation (how we describe rotational motion) and the concept of **moment of inertia** (the rotational analogue of mass). Chapter 12 then builds the **dynamics** (torque, Newton's second law for rotation, and rolling motion).

Linear	**Rotational**	**Relation**
Position x	Angle θ	$s = r\theta$
Velocity v	Angular velocity ω	$v = r\omega$
Acceleration a	Angular acceleration α	$a_{\tan} = r\alpha$
Mass m	Moment of inertia I	$I = \int r_\perp^2 \, dm$
Force F	Torque τ	$\tau = r_\perp F$
Momentum $p = mv$	Angular momentum $L = I\omega$	
Kinetic energy $\frac{1}{2}mv^2$	Rotational KE $\frac{1}{2}I\omega^2$	
$F = ma$	$\tau = I\alpha$	

11.1 Rotational Kinematics

Angular Variables

For rotation about a fixed axis, the state of a rigid body is described by a single variable—the **angular position** $\theta(t)$, measured in radians from some reference line.

Definition 11.1: Angular Kinematic Variables

Angular position: $\theta(t)$ (radians). One complete revolution corresponds to $\theta = 2\pi$ rad $= 360°$.

Angular velocity: $\omega = \dfrac{d\theta}{dt}$ (rad/s). Related to frequency by $\omega = 2\pi f$ and to period by $\omega = 2\pi/T$. Positive ω means counterclockwise (by the right-hand rule).

Angular acceleration: $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$ (rad/s^2).

Key Point 11.1: Radians Are Dimensionless

The radian is defined as arc length divided by radius: $\theta = s/r$. Since both s and r have dimensions of length, the radian is *dimensionless*. This is why we can write $v = r\omega$ without a conversion factor: $[r\omega] = \text{m} \cdot \text{rad/s} = \text{m/s}$ (the "rad" drops out). However, we keep "rad" in units as a reminder that we are describing rotation.

Constant Angular Acceleration

For constant α, the kinematic equations mirror the linear case exactly:

$$\omega = \omega_0 + \alpha t, \tag{11.1}$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2, \tag{11.2}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0). \tag{11.3}$$

These follow from the same calculus as the linear kinematic equations (replace $v \to \omega$, $x \to \theta$, $a \to \alpha$), and are derived identically: integrate $\alpha = \text{const}$ once to get $\omega(t)$, again to get $\theta(t)$, and eliminate t between the first two to get the third.

Example 11.1 (Spinning up a wheel). A grinding wheel ($R = 0.15\,\text{m}$) accelerates from rest to 1200 rpm in 6.0 s with constant angular acceleration. Find (a) α, (b) the number of revolutions, (c) the tangential and centripetal accelerations of a rim point at $t = 6\,\text{s}$.

Solution. First convert: $\omega_f = 1200 \times 2\pi/60 = 40\pi \approx 125.7\,\text{rad/s}$.

(a) $\alpha = (\omega_f - \omega_0)/t = 125.7/6.0 = 20.9\,\text{rad/s}^2$.

(b) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}(20.9)(36) = 377\,\text{rad} = 377/(2\pi) = 60.0$ revolutions.

(c) At $t = 6$ s: $a_{\tan} = r\alpha = 0.15(20.9) = 3.14\,\text{m/s}^2$. $a_c = r\omega^2 = 0.15(125.7)^2 = 2370\,\text{m/s}^2$. The centripetal acceleration vastly exceeds the tangential: the point is almost in uniform circular motion, but not quite.

Example 11.2 (Multi-phase rotation). A flywheel accelerates from rest at $\alpha = 3.0\,\text{rad/s}^2$ for 8.0 s, rotates at constant ω for 5.0 s, then decelerates uniformly to rest in 10.0 s. Find the total angle turned.

Solution. Phase 1 (acceleration): $\omega_1 = 3.0(8.0) = 24\,\text{rad/s}$. $\theta_1 = \frac{1}{2}(3.0)(64) = 96\,\text{rad}$.

Phase 2 (constant ω): $\theta_2 = 24(5.0) = 120\,\text{rad}$.

Phase 3 (deceleration): $\alpha_3 = -24/10 = -2.4\,\text{rad/s}^2$. $\theta_3 = 24(10) + \frac{1}{2}(-2.4)(100) = 240 - 120 = 120\,\text{rad}$.

Total: $\theta = 96 + 120 + 120 = 336\,\text{rad} = 53.5$ revolutions.

Connection to Linear Variables

Every point on a rotating rigid body traces a circular arc. For a point at distance r from the rotation axis:

$$\boxed{s = r\theta, \qquad v_{\tan} = r\omega, \qquad a_{\tan} = r\alpha, \qquad a_c = r\omega^2 = \frac{v_{\tan}^2}{r}.} \tag{11.4}$$

Points farther from the axis move faster (larger $v_{\tan}$) even though all points share the same ω. This is why the outer edge of a spinning disk moves faster than points near the center.

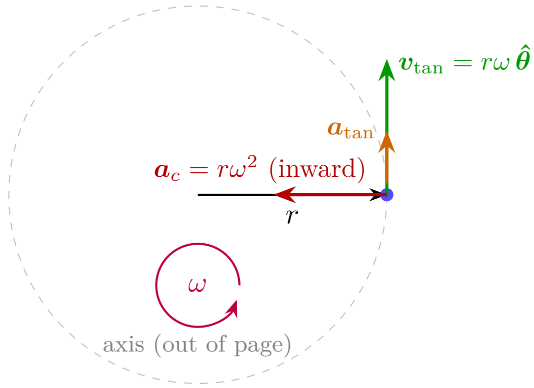


Figure 11.1.1: Velocity and acceleration vectors for a point on a rotating rigid body. $\boldsymbol{v}_{\tan}$ is perpendicular to the radius; $\boldsymbol{a}_c$ points inward; $\boldsymbol{a}_{\tan}$ is parallel to $\boldsymbol{v}$.

Common Mistake 11.1: These Relations Require Constant r

The relations $s = r\theta$, $v_{\tan} = r\omega$, $a_{\tan} = r\alpha$ apply only when each mass element stays at constant distance r from the axis (rigid-body rotation). For general motion (spirals, deformable bodies), the full polar-coordinate expressions from Chapter 4 must be used.

Angular Velocity as a Vector

For rotation about a fixed axis, it is convenient to treat ω as a scalar (positive = counterclockwise). More generally, angular velocity is a *vector* $\boldsymbol{\omega}$ directed along the rotation axis, with the direction given by the right-hand rule: curl the fingers of the right hand in the direction of rotation, and the thumb points along $\boldsymbol{\omega}$. The magnitude is $|\boldsymbol{\omega}| = \omega$.

The velocity of a point at position $\boldsymbol{r}$ from the axis is then:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}. \tag{11.5}$$

This compact formula automatically gives $|\boldsymbol{v}| = \omega r \sin\phi = \omega r_\perp$ (where ϕ is the angle between $\boldsymbol{\omega}$ and $\boldsymbol{r}$) and the correct direction (tangent to the circle).

11.2 Moment of Inertia

Definition and Physical Meaning

The moment of inertia quantifies an object's resistance to changes in rotational motion, playing exactly the same role for rotation that mass plays for translation.

Definition 11.2: Moment of Inertia

For a body rotating about a given axis, the **moment of inertia** about that axis is:

$$\boxed{I = \int r_\perp^2 \, \mathrm{d}m,} \tag{11.6}$$

where $r_\perp$ is the perpendicular distance from each mass element $\mathrm{d}m$ to the rotation axis. For a system of discrete particles: $I = \sum_i m_i r_{i,\perp}^2$.

The key features:

- I depends on the *axis of rotation*—the same object has different moments of inertia about different axes.
- I depends on how mass is *distributed* relative to the axis—mass far from the axis contributes more (because of the r^2 weighting).
- I is always positive and has SI units of $\mathrm{kg\,m^2}$.
- Two objects with the same mass can have very different moments of inertia (a solid sphere vs. a thin hoop of the same mass and radius).

The mass element $\mathrm{d}m$ depends on geometry: $\mathrm{d}m = \rho\,\mathrm{d}V$ (3D body), $\mathrm{d}m = \sigma\,\mathrm{d}A$ (thin plate), or $\mathrm{d}m = \lambda\,\mathrm{d}\ell$ (thin rod/wire).

Calculating Moments of Inertia

Derivation: Uniform disk about the symmetry axis. A disk of mass M, radius R, surface density $\sigma = M/(\pi R^2)$. Decompose into concentric rings of radius r and width $\mathrm{d}r$:

$$\mathrm{d}m = \sigma(2\pi r\,\mathrm{d}r) = \frac{2M}{R^2} r\,\mathrm{d}r.$$

Each ring is entirely at distance r from the axis, so:

$$I = \int_0^R r^2\,\mathrm{d}m = \int_0^R r^2 \cdot \frac{2M}{R^2} r\,\mathrm{d}r = \frac{2M}{R^2} \cdot \frac{R^4}{4} = \boxed{\frac{1}{2}MR^2.}$$

Derivation: Uniform thin rod about its center. A rod of mass M, length L, with axis perpendicular to the rod at its midpoint. Linear density $\lambda = M/L$. Let x be measured from the center:

$$I = \int_{-L/2}^{L/2} x^2\,\lambda\,\mathrm{d}x = \frac{M}{L} \cdot \frac{x^3}{3}\bigg|_{-L/2}^{L/2} = \frac{M}{L} \cdot \frac{2(L/2)^3}{3} = \boxed{\frac{1}{12}ML^2.}$$

Derivation: Uniform solid sphere about a diameter. Slice the sphere into thin disks perpendicular to the axis. A disk at height z above the center has radius $r = \sqrt{R^2 - z^2}$, thickness $\mathrm{d}z$, and mass $\mathrm{d}m = \rho\pi r^2\,\mathrm{d}z$. Each disk has $I_{\text{disk}} = \frac{1}{2}r^2\,\mathrm{d}m$:

$$I = \int_{-R}^{R} \frac{1}{2}\rho\pi(R^2 - z^2)^2\,\mathrm{d}z = \frac{\rho\pi}{2}\int_{-R}^{R}(R^4 - 2R^2z^2 + z^4)\,\mathrm{d}z = \frac{\rho\pi}{2}\left[2R^5 - \frac{4R^5}{3} + \frac{2R^5}{5}\right] = \frac{8\pi\rho R^5}{15}.$$

With $M = \frac{4}{3}\pi\rho R^3$: $\rho = 3M/(4\pi R^3)$, giving $I = \boxed{\frac{2}{5}MR^2}$.

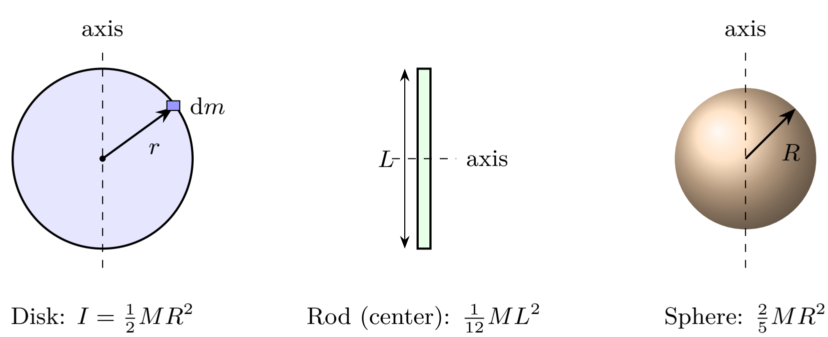


Figure 11.2.1: Moments of inertia for three common objects about their symmetry axes.

Table of Common Moments of Inertia

Object	Axis	I
Thin rod (M, L)	Center, $\perp$ to rod	$\frac{1}{12}ML^2$
Thin rod	One end, $\perp$ to rod	$\frac{1}{3}ML^2$
Uniform disk/cylinder (M, R)	Symmetry axis	$\frac{1}{2}MR^2$
Thin hoop/ring (M, R)	Symmetry axis, $\perp$ to plane	MR^2
Solid sphere (M, R)	Any diameter	$\frac{2}{5}MR^2$
Thin spherical shell (M, R)	Any diameter	$\frac{2}{3}MR^2$
Uniform rectangular plate (M, $a \times b$)	Center, $\perp$ to plate	$\frac{1}{12}M(a^2+b^2)$
Solid cone (M, base R)	Symmetry axis	$\frac{3}{10}MR^2$

Non-Uniform Mass Distributions

Example 11.3 (Non-uniform disk). A thin disk of radius R has surface mass density $\sigma(r) = \sigma_0(r/R)^2$. Find M and I about the center axis.

Solution. $M = \int_0^R \sigma_0(r/R)^2 \cdot 2\pi r\,\mathrm{d}r = \frac{2\pi\sigma_0}{R^2}\int_0^R r^3\,\mathrm{d}r = \frac{2\pi\sigma_0}{R^2}\cdot\frac{R^4}{4} = \frac{\pi\sigma_0 R^2}{2}$.

$I = \int_0^R r^2 \cdot \sigma_0(r/R)^2 \cdot 2\pi r\,\mathrm{d}r = \frac{2\pi\sigma_0}{R^2}\int_0^R r^5\,\mathrm{d}r = \frac{2\pi\sigma_0}{R^2}\cdot\frac{R^6}{6} = \frac{\pi\sigma_0 R^4}{3}$.

Expressing in terms of M: $I = \frac{2}{3}MR^2$. Compare to a uniform disk ($I = \frac{1}{2}MR^2$): the non-uniform disk has a *larger* moment of inertia because the $\sigma \propto r^2$ density concentrates more mass at larger radii, where the r^2 weighting in I amplifies its contribution.

11.3 The Parallel Axis Theorem

Computing I from scratch for every axis is tedious. The parallel axis theorem lets us find I about *any* axis from I about a parallel axis through the center of mass.

Theorem 11.1: Parallel Axis Theorem (Steiner's Theorem)

If I_{cm} is the moment of inertia about an axis through the center of mass, then the moment of inertia about a parallel axis displaced by distance d is:

$$\boxed{I = I_{\text{cm}} + Md^2.} \tag{11.7}$$

Proof. Place the origin at the center of mass. Let $\boldsymbol{r}_i$ be the position of mass element i relative to the CM, and let the new axis be displaced by $\boldsymbol{d}$ from the CM axis. The distance from element i to the new axis involves $\boldsymbol{r}_i + \boldsymbol{d}$ (projected perpendicular to the axis):

$$I = \int |\boldsymbol{r}+\boldsymbol{d}|^2\,\mathrm{d}m = \int |\boldsymbol{r}|^2\,\mathrm{d}m + 2\boldsymbol{d}\cdot\int \boldsymbol{r}\,\mathrm{d}m + |\boldsymbol{d}|^2\int \mathrm{d}m.$$

The first integral is I_{cm}. The second integral vanishes: $\int \boldsymbol{r}\,\mathrm{d}m = M\boldsymbol{R}_{\text{cm}} = \boldsymbol{0}$ (since the origin is at the CM). The third integral is Md^2. Therefore $I = I_{\text{cm}} + Md^2$.

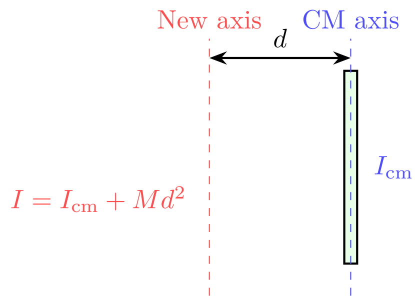


Figure 11.3.1: The parallel axis theorem: the moment of inertia about any axis equals I_{cm} plus Md^2.

Common Mistake 11.2: The Parallel Axis Theorem Only Works from the CM

The theorem states $I = I_{\text{cm}} + Md^2$—one of the two axes *must* pass through the center of mass. You cannot use it to relate moments of inertia about two arbitrary parallel axes that both miss the CM. (To do so, apply the theorem twice: go from axis 1 to the CM axis, then from the CM axis to axis 2.)

Worked Examples

Example 11.4 (Rod about one end). $I_{\text{cm}} = \frac{1}{12}ML^2$ (center of rod). The end is at distance $d = L/2$ from the CM. By the parallel axis theorem:

$$I_{\text{end}} = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2.$$

Example 11.5 (Sphere about a tangent line). $I_{\text{cm}} = \frac{2}{5}MR^2$ (diameter). A tangent line is at distance $d = R$ from the center.

$$I_{\text{tangent}} = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2.$$

Example 11.6 (Minimizing I for a rod). A rod of mass M and length L is pivoted at a point a distance x from one end. The CM is at $L/2$, so $d = |x - L/2|$.

$$I(x) = \frac{1}{12}ML^2 + M(x - L/2)^2.$$

This is minimized when $d = 0$, i.e., $x = L/2$ (pivot at the center), giving $I_{\text{min}} = \frac{1}{12}ML^2$.

11.4 The Perpendicular Axis Theorem

Theorem 11.2: Perpendicular Axis Theorem (Planar Objects Only)

For a flat (planar) object lying in the xy-plane:

$$\boxed{I_z = I_x + I_y.} \tag{11.8}$$

Proof. For a mass element at $(x, y, 0)$: $I_z = \int (x^2 + y^2)\,\mathrm{d}m = \int y^2\,\mathrm{d}m + \int x^2\,\mathrm{d}m = I_x + I_y$.

This theorem fails for 3D objects because a mass element at (x, y, z) with $z \neq 0$ contributes $x^2 + y^2$ to I_z but $y^2 + z^2$ to I_x and $x^2 + z^2$ to I_y, so $I_x + I_y = x^2 + y^2 + 2z^2 \neq I_z$.

Example 11.7 (Thin uniform disk about a diameter). By symmetry, $I_x = I_y$ (both are diameters). Using $I_z = \frac{1}{2}MR^2$ (the standard result for the axis perpendicular to the disk):

$$I_z = I_x + I_y = 2I_x \quad \Longrightarrow \quad I_x = \frac{1}{4}MR^2.$$

This is a far easier route than integrating directly.

Example 11.8 (Thin square plate about the z-axis). A uniform square plate of mass M and side a lies in the xy-plane with center at the origin and sides parallel to the axes. By symmetry, $I_x = I_y$. Each is the moment of inertia about an axis along a line of symmetry parallel to one side. Direct integration (or recognizing the plate as a stack of thin rods) gives $I_x = Ma^2/12$. Therefore:

$$I_z = I_x + I_y = \frac{Ma^2}{12} + \frac{Ma^2}{12} = \frac{Ma^2}{6}.$$

11.5 Rotational Kinetic Energy

Definition 11.3: Rotational Kinetic Energy

A rigid body rotating about a fixed axis with angular velocity ω has rotational kinetic energy:

$$\boxed{K_{\text{rot}} = \tfrac{1}{2}I\omega^2.} \tag{11.9}$$

Derivation. Each mass element $\mathrm{d}m$ at distance $r_\perp$ from the axis has speed $v = r_\perp\omega$. Summing:

$$K = \int \tfrac{1}{2}v^2 \,\mathrm{d}m = \int \tfrac{1}{2}r_\perp^2\omega^2 \,\mathrm{d}m = \tfrac{1}{2}\omega^2 \int r_\perp^2 \,\mathrm{d}m = \tfrac{1}{2}I\omega^2.$$

Total Kinetic Energy: Translation + Rotation

For an object that both translates and rotates (e.g., a rolling wheel), the total kinetic energy decomposes via König's theorem (Theorem 10.3, proved in Section 10.6):

$$\boxed{K_{\text{total}} = \tfrac{1}{2}Mv_{\text{cm}}^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2.} \tag{11.10}$$

The first term is the translational KE of the center of mass; the second is the rotational KE about the center of mass. These two contributions are independent: the total KE is their sum, with no cross terms.

Example 11.9 (Rolling hoop). A bicycle wheel (thin hoop, $M = 1.5\,\text{kg}$, $R = 0.35\,\text{m}$) rolls without slipping at $v = 7.0\,\text{m/s}$ ($\omega = v/R = 20\,\text{rad/s}$).

$K_{\text{trans}} = \frac{1}{2}Mv^2 = \frac{1}{2}(1.5)(49) = 36.75\,\text{J}$.

$K_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2}(MR^2)(v/R)^2 = \frac{1}{2}Mv^2 = 36.75\,\text{J}$.

$K_{\text{total}} = 36.75 + 36.75 = 73.5\,\text{J}$.

For a rolling hoop, exactly half the kinetic energy is translational and half is rotational. For a rolling solid cylinder ($I = \frac{1}{2}MR^2$), the split is 2/3 translational, 1/3 rotational. For a rolling solid sphere ($I = \frac{2}{5}MR^2$), it is 5/7 translational, 2/7 rotational.

Work-Energy Theorem for Rotation

The rotational analogue of $W = \int F\,\mathrm{d}x = \Delta K_{\text{trans}}$ is:

$$W = \int \tau\,\mathrm{d}\theta = \Delta K_{\text{rot}} = \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2. \tag{11.11}$$

For constant torque: $W = \tau\,\Delta\theta$. The rotational power is $P = \tau\omega$ (analogous to $P = Fv$).

Example 11.10 (Rotational work-energy). A wheel ($I = 0.50\,\text{kg}\,\text{m}^2$, $R = 0.20\,\text{m}$) has a tangential force of 10 N applied at the rim. Starting from rest, find ω and K_{rot} after 4 s, and verify using $W = \tau\Delta\theta$.

Solution. $\tau = FR = 10(0.20) = 2.0\,\text{N}\,\text{m}$. $\alpha = \tau/I = 2.0/0.50 = 4.0\,\text{rad/s}^2$.

After 4 s: $\omega = \alpha t = 16$ rad/s. $K_{\text{rot}} = \frac{1}{2}(0.50)(256) = 64\,\text{J}$.

Angle: $\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2}(4.0)(16) = 32\,\text{rad}$. Work: $W = \tau\theta = 2.0(32) = 64\,\text{J}$. ✓

Problems

Problem 11.1 ⋆

A wheel of radius 0.30 m makes 120 revolutions per minute. Find (a) the angular velocity in rad/s, (b) the speed of a point on the rim, (c) the centripetal acceleration of a point on the rim.

Problem 11.2 ⋆⋆

A grinding wheel ($R = 0.15\,\text{m}$) accelerates from rest to 1200 rpm in 6.0 s. (a) Find α. (b) Total angle in the first 6.0 s. (c) Tangential and centripetal accelerations of a rim point at $t = 6.0\,\text{s}$.

Problem 11.3 ⋆⋆

A uniform solid sphere of mass M and radius R rotates about an axis tangent to its surface. Find I using the parallel axis theorem.

Problem 11.4 ⋆⋆⋆

A thin disk of radius R has surface mass density $\sigma(r) = \sigma_0(r/R)^2$. (a) Find M. (b) Find I about the center axis. (c) Compare to a uniform disk and explain physically.

Problem 11.5 ⋆⋆⋆

A uniform thin square plate of mass M and side a lies in the xy-plane. (a) Show that $I_x = I_y = Ma^2/12$. (b) Use the perpendicular axis theorem to find I_z. (c) Use the parallel axis theorem to find I about an axis along one edge of the square.

Problem 11.6 ⋆⋆⋆

A wheel has moment of inertia $I = 0.50\,\text{kg m}^2$. A tangential force of 10 N is applied at the rim ($R = 0.20\,\text{m}$). Starting from rest, find (a) α, (b) ω after 4.0 s, (c) the rotational KE after 4.0 s, (d) verify using $W = \tau\Delta\theta$.

Problem 11.7 ⋆⋆⋆

A bicycle wheel ($M = 1.5\,\text{kg}$, $R = 0.35\,\text{m}$, modeled as a thin hoop) spins at $\omega = 20\,\text{rad/s}$. (a) Find its rotational KE. (b) If the wheel also rolls without slipping ($v = R\omega$), find the total KE. (c) What fraction of the total KE is rotational?

Problem 11.8 ⋆⋆⋆⋆

Derive the moment of inertia of a uniform solid sphere ($I = \frac{2}{5}MR^2$) about a diameter by slicing into thin disks perpendicular to the axis.

Problem 11.9 ⋆⋆⋆⋆

A uniform rod of mass M and length L is pivoted at a point a distance x from one end. (a) Find $I(x)$. (b) Determine the value of x that minimizes I. (c) For $x = 0$ (pivot at one end), compare I to the center-pivot result and explain the difference physically.

Problem 11.10 ⋆⋆⋆⋆⋆

A uniform cone of mass M, height h, and base radius R has its apex at the origin and base at height h. Derive the moment of inertia about the symmetry axis: $I = \frac{3}{10}MR^2$.

Chapter 12

Dynamics of Rotational Motion

In Chapter 11 we developed the *kinematics* of rotation (angular variables, moment of inertia, rotational kinetic energy). In this chapter we develop the *dynamics*: what *causes* rotational motion and how it evolves. The central result is the rotational analogue of Newton's second law, $\sum \tau = I\alpha$, and we apply it to pulleys, rolling objects, and angular-impulse problems.

12.1 Torque

Definition and Geometry

Torque is the rotational analogue of force. Just as a force causes linear acceleration, a torque causes angular acceleration.

Definition 12.1: Torque

The torque about a point O due to a force $\boldsymbol{F}$ applied at position $\boldsymbol{r}$ relative to O is

$$\boxed{\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}.} \tag{12.1}$$

The magnitude is $\tau = rF\sin\phi = r_\perp F = rF_\perp$, where ϕ is the angle between $\boldsymbol{r}$ and $\boldsymbol{F}$, $r_\perp = r\sin\phi$ is the **moment arm** (lever arm), and $F_\perp = F\sin\phi$ is the perpendicular component of the force. The SI unit is N m.

There are two equivalent ways to compute a torque's magnitude:

- $\tau = r_\perp F$: the full force times the perpendicular distance from the pivot to the force's line of action.
- $\tau = rF_\perp$: the full distance times the force component perpendicular to $\boldsymbol{r}$.

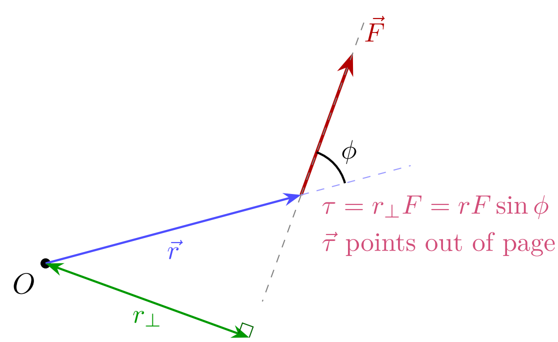


Figure 12.1.1: Torque geometry: the moment arm $r_\perp$ is the perpendicular distance from the pivot O to the line of action of $\boldsymbol{F}$.

Key Point 12.1: Torque Depends on the Pivot

Unlike force, torque is defined *relative to a chosen point.* The same force produces different torques about different points. A force whose line of action passes through the pivot produces zero torque ($r_\perp = 0$). When multiple forces act, the net torque is $\tau_{\text{net}} = \sum_i \tau_i$, all computed about the *same* point.

Sign convention. For rotation in a plane, counterclockwise torques are positive and clockwise torques are negative (right-hand rule: $\boldsymbol{\tau}$ out of the page for CCW).

Example 12.1 (Wrench). A mechanic applies 80 N at the end of a 0.30 m wrench at 90° to the handle: $\tau = 0.30(80)(1) = 24\,\text{N m}$. At 60°: $\tau = 0.30(80)\sin 60° = 20.8\,\text{N m}$—less effective because part of the force pulls along the wrench rather than rotating it.

12.2 Newton's Second Law for Rotation

Theorem 12.1: Rotational Newton's Second Law

For a rigid body rotating about a fixed axis with moment of inertia I:

$$\boxed{\sum \tau = I\alpha.} \tag{12.2}$$

Derivation

Starting from $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ for a particle:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}} = \underbrace{\boldsymbol{v} \times m\boldsymbol{v}}_{=\boldsymbol{0}} + \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\tau}.$$

For a rigid body about a fixed axis with $L = I\omega$: $\tau = \mathrm{d}(I\omega)/\mathrm{d}t = I\alpha$ (since I is constant).

Key Point 12.2: Choosing the Pivot

All torques and I must be computed about the *same* point. Choosing a point where an unknown force acts eliminates that force from the torque equation. The two "safe" choices: (1) a fixed point in the lab frame (hinge, axle), or (2) the center of mass (works even if the CM accelerates).

Strategy 12.1: Rotational Dynamics Problems

1. **Draw an FBD** showing all forces and their points of application.
2. **Choose a pivot**—one that eliminates unknown forces.
3. **Write** $\sum \tau = I\alpha$ (CCW = positive).
4. **Write** $\sum F = ma$ for any translational motion.
5. **Apply constraints** ($a = R\alpha$ for strings on pulleys; $v = R\omega$ for rolling).
6. **Solve** the system of equations.

Worked Examples

Example 12.2 (Pulley with mass). A block of mass m hangs from a string wrapped around a disk pulley of mass M and radius R. Find the acceleration.

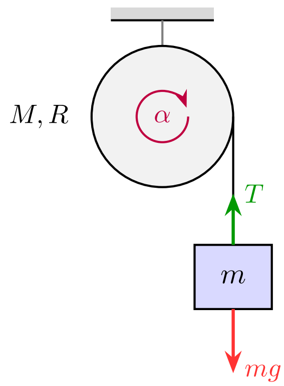


Figure 12.2.1: A massive disk pulley with a hanging block.

Solution. Block: $mg - T = ma$. Pulley: $TR = \frac{1}{2}MR^2(a/R)$, so $T = \frac{1}{2}Ma$. Substituting:

$$a = \frac{mg}{m + M/2}, \qquad T = \frac{mMg}{2m + M}.$$

If $M = 0$: $a = g$, $T = 0$. If $M \gg m$: $a \to 0$. Energy check: after descending h, $mgh = \frac{1}{2}(m+M/2)v^2$, confirming $a = mg/(m + M/2)$. ✓

Example 12.3 (Atwood with massive pulley). Masses m_1, $m_2 > m_1$ over a solid disk pulley of mass M, radius R. String doesn't slip.

Solution. Key: $T_1 \neq T_2$—the difference provides the pulley's torque.

Block 1: $T_1 - m_1 g = m_1 a$. Block 2: $m_2 g - T_2 = m_2 a$. Pulley: $(T_2 - T_1)R = \frac{1}{2}MR^2(a/R)$, so $T_2 - T_1 = \frac{1}{2}Ma$. Adding:

$$\boxed{a = \frac{(m_2 - m_1)g}{m_1 + m_2 + M/2}.}$$

Common Mistake 12.1: Tensions Differ on a Massive Pulley

On a pulley with $I \neq 0$, the tensions must differ—otherwise no net torque and no angular acceleration. Only for $I = 0$ (massless pulley) are they equal.

Example 12.4 (Falling rod). A uniform rod (M, L) pivoted at one end is released from horizontal. Find the initial α, $\omega(\theta)$, and the tip speed when vertical.

Solution. $I = \frac{1}{3}ML^2$. Torque: $\tau = Mg(L/2)$ at $\theta = 0$. $\alpha_0 = 3g/(2L)$.

Energy (CM falls $(L/2)\sin\theta$): $\frac{1}{2}I\omega^2 = Mg(L/2)\sin\theta$, so $\omega = \sqrt{3g\sin\theta/L}$.

At $\theta = 90°$: $v_{\text{tip}} = L\omega = \sqrt{3gL}$. This exceeds $\sqrt{2gL}$ (a point mass falling distance L)—the pivot constrains the upper part of the rod to "push" the lower part faster than free fall.

12.3 Rolling Without Slipping

Definition 12.2: Rolling Constraint

An object of radius R rolls without slipping when the contact point has zero velocity:

$$\boxed{v_{\text{cm}} = R\omega, \qquad a_{\text{cm}} = R\alpha.} \tag{12.3}$$

In one full rotation the center advances $2\pi R$—exactly one circumference.

Velocities on a Rolling Object

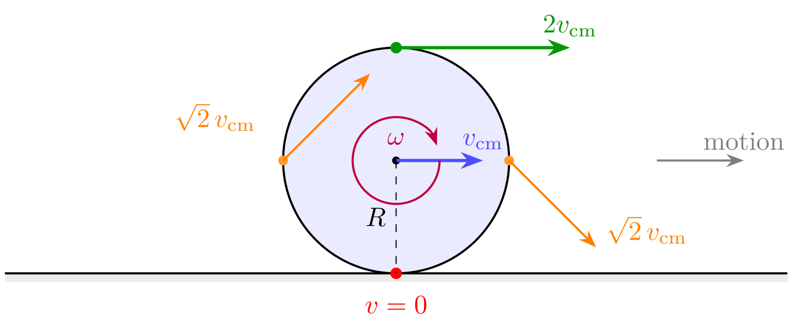


Figure 12.3.1: Velocities on a rolling object: $v = 0$ at contact, v_{cm} at center, $2v_{\text{cm}}$ at top.

$\boldsymbol{v}_P = \boldsymbol{v}_{\text{cm}} + \boldsymbol{v}_{\text{rot}}$. At the contact point, rotation gives $R\omega$ backward, canceling v_{cm} forward. At the top, rotation adds $R\omega$ forward: $v_{\text{top}} = 2v_{\text{cm}}$.

Key Point 12.3: Friction in Rolling

During rolling without slipping, friction is *static*—the contact point has zero velocity. Static friction can point forward or backward (determined by the equations of motion), and it does *no work* because the point of application has zero displacement.

Energy of Rolling

Using $v_{\text{cm}} = R\omega$ and $I_{\text{cm}} = cMR^2$:

$$K = \tfrac{1}{2}Mv_{\text{cm}}^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2 = \tfrac{1}{2}Mv_{\text{cm}}^2(1 + c). \tag{12.4}$$

Object	$c = I/(MR^2)$	**Fraction rotational**
Solid sphere	2/5	$2/7 \approx 29\%$
Solid cylinder	1/2	$1/3 \approx 33\%$
Thin spherical shell	2/3	$2/5 = 40\%$
Thin hoop	1	$1/2 = 50\%$

Rolling Down an Incline

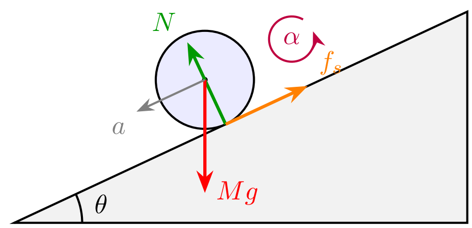


Figure 12.3.2: FBD for a body rolling down an incline without slipping.

Newton's-law derivation. Along the incline: $Mg\sin\theta - f_s = Ma$. Rotation about CM: $f_sR = cMR^2(a/R) = cMRa$, so $f_s = cMa$. Substituting:

$$\boxed{a = \frac{g\sin\theta}{1+c}, \qquad v = \sqrt{\frac{2gh}{1+c}}.} \tag{12.5}$$

Energy derivation. $Mgh = \frac{1}{2}Mv^2(1+c)$, same result. Objects with smaller c accelerate faster; the result is independent of M and R.

Minimum friction for rolling. $f_s = cMg\sin\theta/(1+c)$ and $N = Mg\cos\theta$, so $f_s \le \mu_s N$ requires:

$$\mu_s \ge \frac{c\tan\theta}{1+c}. \tag{12.6}$$

Key Point 12.4: Without Enough Friction

If μ_s is too small, the object slides while rotating. Kinetic friction applies, $v_{\text{cm}} \neq R\omega$, and translation and rotation decouple.

Rolling on Moving Surfaces

On a surface moving at V, the rolling constraint becomes $v_{\text{cm}} - V = R\omega$; this ensures that the relative contact-point velocity is zero.

12.4 Transition from Sliding to Rolling

Example 12.5 (Bowling ball). A solid ball (M, R) is placed on a horizontal surface at speed v_0 with zero spin. Kinetic friction μ_k.

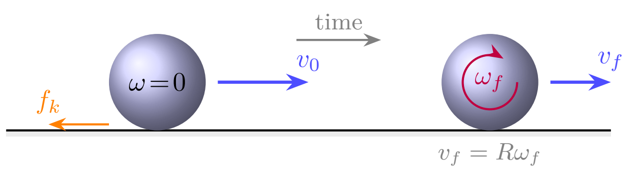


Figure 12.4.1: Sliding → rolling transition. Friction decelerates translation while spinning up rotation until $v = R\omega$.

Solution. Translation: $v(t) = v_0 - \mu_k gt$. Rotation about CM: $\alpha = 5\mu_k g/(2R)$, so $\omega(t) = 5\mu_k g\, t/(2R)$.

Rolling at $v = R\omega$: $v_0 - \mu_k g\, t_r = \frac{5}{2}\mu_k g\, t_r$, giving $t_r = 2v_0/(7\mu_k g)$ and $v_f = 5v_0/7$.

Energy lost: $\Delta K/K_0 = 2/7 \approx 29\%$.

Key Point 12.5: The 5/7 Rule

A solid sphere sliding at v_0 with no spin always reaches rolling at $v_f = 5v_0/7$, independent of μ_k. This also follows from angular-momentum conservation about the contact point (where friction has zero lever arm): $Mv_0R = \frac{7}{5}Mv_fR$, giving $v_f = 5v_0/7$.

12.5 More Worked Examples

Example 12.6 (Race down an incline). A solid sphere ($c = 2/5$) and a solid cylinder ($c = 1/2$) roll from rest. $a_{\text{sphere}} = 5g\sin\theta/7$, $a_{\text{cyl}} = 2g\sin\theta/3$. Since $5/7 > 2/3$, the sphere wins.

Example 12.7 (Yo-yo). Solid disk on a string at the rim. Translation: $Mg - T = Ma$. Rotation: $T = \frac{1}{2}Ma$. So $a = 2g/3$, $T = Mg/3$. If wound around a thin axle of radius $r \ll R$: $a = g/(1 + R^2/(2r^2)) \to 0$.

Example 12.8 (Angular impulse). A ball (m, v_0) hits and sticks to the bottom of a hanging rod (M, L, pivoted at top). Angular momentum about pivot is conserved: $mv_0L = (\frac{1}{3}ML^2 + mL^2)\omega$, so $\omega = 3mv_0/((M + 3m)L)$. Linear momentum is *not* conserved (the pivot exerts an impulse).

Example 12.9 (Turntable). Disk: $M = 5.0$ kg, $R = 0.25$ m, rest to 33 rpm in 4.0 s. $I = 0.156$ kg·m^2, $\omega_f = 3.46$ rad/s, $\alpha = 0.864$ rad/s^2. $\tau = 0.135$ N·m. $W = \tau\theta = 0.135(6.91) = 0.93$ J $= \frac{1}{2}I\omega_f^2$. ✓

Problems

Problem 12.1 ⋆⋆

A flywheel starts from rest and undergoes three phases of motion: it accelerates uniformly at $\alpha = 3.0\,\text{rad/s}^2$ for 8.0 s, then rotates at constant angular velocity for 5.0 s, then decelerates uniformly to rest in 10.0 s. (a) Find the angular velocity at the end of the acceleration phase. (b) Find the total angle turned through during all three phases. (c) Find the angular deceleration during the third phase.

Problem 12.2 ⋆⋆⋆

A thin disk of radius R has a non-uniform surface mass density $\sigma(r) = \sigma_0(r/R)^2$, where r is the distance from the center. (a) Find the total mass M in terms of σ_0 and R. (b) Find the moment of inertia about the central axis perpendicular to the disk, expressed in terms of M and R. (c) Compare to a uniform disk of the same mass and radius, and explain physically why the result differs.

Problem 12.3 ⋆⋆⋆⋆

An Atwood machine consists of two blocks of masses m_1 and $m_2 > m_1$ connected by a light string that passes over a uniform solid disk pulley of mass M and radius R. The string does not slip on the pulley. (a) Explain why the tensions on the two sides of the string must be different. (b) Using Newton's second law for both blocks and $\tau = I\alpha$ for the pulley, derive $a = (m_2 - m_1)g/(m_1 + m_2 + M/2)$. (c) Find the tensions T_1 and T_2 on each side of the string. (d) Use energy conservation to find the speed of the blocks after m_2 descends a height h from rest. Verify consistency with part (b). (e) Compute numerical values for $m_1 = 2\,\text{kg}$, $m_2 = 5\,\text{kg}$, $M = 4\,\text{kg}$, $h = 1.5\,\text{m}$.

Problem 12.4 ⋆⋆⋆⋆

A uniform solid cylinder of mass M and radius R is released from rest at the top of a rough incline of angle θ and length L. At the bottom, it rolls onto a second incline of angle ϕ that is *frictionless*. (a) Find the linear acceleration of the cylinder on the rough incline (rolling without slipping). (b) Find the minimum coefficient of static friction μ_s required for rolling without slipping on the rough incline. (c) Find the speed of the cylinder at the bottom of the rough incline. (d) Describe qualitatively what happens to the cylinder on the frictionless incline. Does it continue to roll without slipping? (e) Find the maximum height h reached on the frictionless incline. Explain why it is less than the starting height $L\sin\theta$. (f) Compute numerical values for $\theta = 30°$, $\phi = 45°$, $L = 2\,\text{m}$.

Problem 12.5 ⋆⋆⋆

A uniform rod of mass M and length L is pivoted at one end and released from a horizontal position. (a) Find the initial angular acceleration α_0 (at the moment of release). (b) Using energy conservation, find the angular velocity ω as a function of the angle θ below the horizontal. (c) Find the angular momentum $L_{\text{ang}}(\theta)$ about the pivot as a function of θ. (d) At what angle θ is the angular momentum maximized? (e) Compute numerical values of α_0, $\omega(90°)$, and $L_{\max}$ for $M = 0.5\,\text{kg}$, $L = 1.2\,\text{m}$.

Problem 12.6 ⋆⋆⋆

A uniform solid sphere of mass M and radius R rolls without slipping down an incline of angle θ and height h. (a) Find the linear acceleration of the center of mass. (b) Find the minimum coefficient of static friction required for rolling without slipping. (c) Find the speed at the bottom of the incline and compare it to (i) a frictionless sliding block and (ii) a rolling thin-walled hollow cylinder of the same mass. Which arrives first?

Problem 12.7 ⋆⋆⋆

A yo-yo is modeled as a uniform solid disk of mass M and radius R. A string is wound around its rim and attached to a fixed support. The yo-yo is released from rest. (a) Find the downward acceleration of the yo-yo. (b) Find the tension in the string. (c) Express the acceleration as a fraction of g and comment on how a yo-yo with a very thin axle (radius $r \ll R$) would behave differently.

Problem 12.8 ⋆⋆⋆⋆

A uniform solid ball of mass M and radius R is placed on a horizontal surface with initial translational velocity v_0 but no spin ($\omega_0 = 0$). The coefficient of kinetic friction between the ball and the surface is μ_k. (a) Write separate equations for the translational and rotational motion during the sliding phase, and find the time t_r at which rolling without slipping begins. (b) Find the final translational velocity v_f and show that $v_f = 5v_0/7$, independent of μ_k. (c) What fraction of the initial kinetic energy is lost to friction during the sliding phase?

Problem 12.9 ⋆⋆⋆

A turntable is modeled as a uniform disk of mass $M = 5.0\,\mathrm{kg}$ and radius $R = 0.25\,\mathrm{m}$. A motor brings it from rest to 33 rpm with constant angular acceleration in 4.0 s. (a) Find the torque exerted by the motor. (b) Find the total work done by the motor. (c) Verify your answer to (b) using the rotational work-energy theorem ($W = \Delta K_{\mathrm{rot}}$).

Chapter 13

Angular Momentum

We have now developed torque, the rotational analogue of force, and $\sum \tau = I\alpha$, the rotational analogue of Newton's second law. In this chapter we complete the parallel by introducing **angular momentum**—the rotational analogue of linear momentum. The payoff is enormous: when the net external torque on a system vanishes, angular momentum is *conserved*, giving us the third great conservation law of mechanics (after energy and linear momentum).

Conservation of angular momentum explains phenomena as diverse as an ice skater's spin, the formation of neutron stars, the stability of gyroscopes, and Kepler's second law of planetary motion.

13.1 Angular Momentum of a Particle

Definition 13.1: Angular Momentum

The angular momentum of a particle with momentum $\boldsymbol{p} = m\boldsymbol{v}$ about a point O is

$$\boxed{\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v},} \tag{13.1}$$

where $\boldsymbol{r}$ is the position vector from O to the particle. The magnitude is $L = rmv\sin\phi = r_\perp p$, where ϕ is the angle between $\boldsymbol{r}$ and $\boldsymbol{v}$, and $r_\perp = r\sin\phi$ is the perpendicular distance from O to the line of motion. The SI unit is $\mathrm{kg\,m^2/s}$ (equivalently, $\mathrm{J\,s}$).

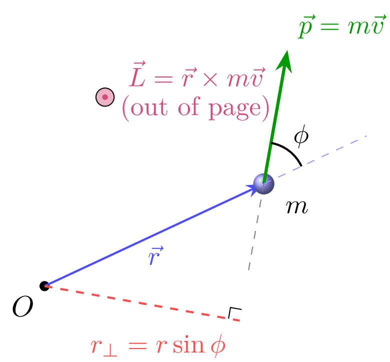


Figure 13.1.1: Angular momentum of a particle: $\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v}$. The magnitude is $L = r_\perp p = rmv\sin\phi$.

Several features parallel the angular quantities from torque:

- Angular momentum, like torque, is defined *relative to a chosen point* O. Different points give different $\boldsymbol{L}$.
- The direction of $\boldsymbol{L}$ is perpendicular to the plane of $\boldsymbol{r}$ and $\boldsymbol{v}$, given by the right-hand rule.
- A particle moving *directly toward or away from* O has $\boldsymbol{L} = \boldsymbol{0}$ about O (since $\sin\phi = 0$).

Example 13.1 (Straight-line motion). A particle of mass m moves along a straight line with constant velocity v at perpendicular distance d from point O. Its angular momentum about O is $L = mvd$ and is *constant*, even though the particle is not moving in a circle. This is because r increases while $\sin\phi$ decreases in exactly compensating fashion: $r\sin\phi = d = \text{const}$.

13.2 Angular Momentum of a Rigid Body

For a rigid body rotating about a fixed axis with angular velocity ω, each mass element $\mathrm{d}m$ at distance $r_\perp$ from the axis has tangential speed $v = r_\perp\omega$. Its angular momentum about the axis is $\mathrm{d}L = r_\perp(v\,\mathrm{d}m) = r_\perp^2\omega\,\mathrm{d}m$. Summing:

$$\boxed{L = I\omega,} \tag{13.2}$$

which is the direct rotational analogue of $p = mv$.

Combined Translation and Rotation

For an object that both translates and rotates (e.g., a rolling wheel or a tumbling wrench), the total angular momentum about a fixed point O decomposes as:

$$\boxed{\boldsymbol{L}_O = I_{\text{cm}}\,\omega\,\hat{\boldsymbol{n}} + \boldsymbol{R}_{\text{cm}} \times M\boldsymbol{v}_{\text{cm}},} \tag{13.3}$$

where $I_{\text{cm}}\omega\,\hat{\boldsymbol{n}}$ is the **spin angular momentum** (rotation about the CM) and $\boldsymbol{R}_{\text{cm}} \times M\boldsymbol{v}_{\text{cm}}$ is the **orbital angular momentum** (motion of the CM about O).

Derivation. Write the position of each mass element as $\boldsymbol{r}_i = \boldsymbol{R}_{\text{cm}} + \boldsymbol{r}_i'$ and its velocity as $\boldsymbol{v}_i = \boldsymbol{v}_{\text{cm}} + \boldsymbol{v}_i'$, where primes denote quantities relative to the CM. Then:

$$\begin{aligned}\boldsymbol{L}_O &= \sum_i m_i\boldsymbol{r}_i \times \boldsymbol{v}_i = \sum_i m_i(\boldsymbol{R}_{\text{cm}} + \boldsymbol{r}_i') \times (\boldsymbol{v}_{\text{cm}} + \boldsymbol{v}_i')\\ &= M(\boldsymbol{R}_{\text{cm}} \times \boldsymbol{v}_{\text{cm}}) + \boldsymbol{R}_{\text{cm}} \times \underbrace{\sum_i m_i\boldsymbol{v}_i'}_{=\boldsymbol{0}} + \underbrace{\sum_i m_i\boldsymbol{r}_i'}_{=\boldsymbol{0}} \times \boldsymbol{v}_{\text{cm}} + \sum_i m_i\boldsymbol{r}_i' \times \boldsymbol{v}_i'.\end{aligned}$$

The two cross terms vanish because $\sum m_i\boldsymbol{r}_i' = \boldsymbol{0}$ and $\sum m_i\boldsymbol{v}_i' = \boldsymbol{0}$ (by definition of the CM). The last term is $\boldsymbol{L}_{\text{spin}} = I_{\text{cm}}\omega\,\hat{\boldsymbol{n}}$ for a rigid body.

This decomposition is the angular-momentum analogue of König's theorem for kinetic energy. It is used constantly in rolling and collision problems: the spin and orbital contributions can be computed independently and added.

For a system of particles sharing a common rotation axis (e.g., a person on a turntable), all spinning together at ω:

$$L_{\text{total}} = (I_1 + I_2 + \cdots)\omega.$$

13.3 Angular Momentum of a Projectile

A projectile in a gravitational field has angular momentum about any point. About the launch point, for a projectile of mass m at position $\boldsymbol{r}$ with velocity $\boldsymbol{v}$:

$$\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} = m(x\dot{y} - y\dot{x})\,\hat{\boldsymbol{k}}.$$

The gravitational torque about the launch point is $\tau = -mgx(t)\,\hat{\boldsymbol{k}} \neq 0$ in general, so L is *not* constant about an arbitrary point for uniform-gravity projectiles.

However, for a **central force**—one directed along $\boldsymbol{r}$, such as gravity from a point mass—the torque about the force center is *always* zero:

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{0} \qquad (\text{because } \boldsymbol{F} \parallel \boldsymbol{r}).$$

Therefore angular momentum about the force center is conserved. This is the foundation of Kepler's second law (Chapter 16).

13.4 The Torque–Angular Momentum Relation

Single Particle

Theorem 13.1: Rotational Analogue of Newton's Second Law

The net torque about a point equals the rate of change of angular momentum about that point:

$$\boxed{\boldsymbol{\tau}_{\text{net}} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}.} \tag{13.4}$$

Proof. Differentiate $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}} = \underbrace{\boldsymbol{v} \times m\boldsymbol{v}}_{=\boldsymbol{0}} + \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\tau}.$$

For a rigid body about a fixed axis with constant I: $\tau = \mathrm{d}(I\omega)/\mathrm{d}t = I\alpha$. But Eq. (13.4) also applies when I changes (e.g., a collapsing star), in which case $\tau = I\dot{\omega} + \dot{I}\omega$—and the simpler $\tau = I\alpha$ does not hold.

System of Particles

For a system of N particles, the total angular momentum about a point O is $\boldsymbol{L}_{\text{total}} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i$. Differentiating:

$$\frac{\mathrm{d}\boldsymbol{L}_{\text{total}}}{\mathrm{d}t} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \sum_i \boldsymbol{r}_i \times (\boldsymbol{F}_i^{\text{ext}} + \sum_{j \neq i} \boldsymbol{F}_{ij}).$$

The internal forces $\boldsymbol{F}_{ij}$ come in Newton's-third-law pairs: $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$, and (crucially) they act along the line joining particles i and j (the strong form of Newton's third law). The torque from such a pair about O is:

$$\boldsymbol{r}_i \times \boldsymbol{F}_{ij} + \boldsymbol{r}_j \times \boldsymbol{F}_{ji} = (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} = \boldsymbol{0},$$

because $\boldsymbol{F}_{ij}$ is parallel to $(\boldsymbol{r}_i - \boldsymbol{r}_j)$. All internal torques cancel in pairs, leaving:

$$\boxed{\boldsymbol{\tau}_{\text{ext}} = \frac{\mathrm{d}\boldsymbol{L}_{\text{total}}}{\mathrm{d}t}.} \tag{13.5}$$

Only *external* torques change the total angular momentum of a system. This is the rotational analogue of $\boldsymbol{F}_{\text{ext}} = \mathrm{d}\boldsymbol{P}/\mathrm{d}t$ for linear momentum, and it is the foundation of angular momentum conservation.

Common Mistake 13.1: The Strong Form of Newton's Third Law

The cancellation of internal torques requires that internal forces act *along the line joining the two particles* (not just equal and opposite). This is the "strong form" of Newton's third law. It holds for all contact forces and for gravity, electrostatics, and other central forces, but it fails for magnetic forces between moving charges. In such cases, the field itself carries angular momentum, and the total (particles + field) is still conserved.

Angular Impulse

Integrating $\boldsymbol{\tau} = \mathrm{d}\boldsymbol{L}/\mathrm{d}t$ over a time interval gives the **angular impulse–angular momentum theorem**:

$$\boxed{\boldsymbol{J}_{\text{ang}} = \int_{t_i}^{t_f} \boldsymbol{\tau}\,\mathrm{d}t = \Delta\boldsymbol{L} = \boldsymbol{L}_f - \boldsymbol{L}_i.} \tag{13.6}$$

This is the rotational analogue of the (linear) impulse-momentum theorem $\boldsymbol{J} = \Delta\boldsymbol{p}$.

For a constant torque: $J_{\text{ang}} = \tau\,\Delta t$. For a brief collision in which a large torque acts over a short time (e.g., a ball striking a pivoted rod), the angular impulse determines ΔL even though the instantaneous torque may be unknown.

Key Point 13.1: Three Parallel Laws

Translation	**Rotation**
$\boldsymbol{F}_{\text{net}} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$	$\boldsymbol{\tau}_{\text{net}} = \mathrm{d}\boldsymbol{L}/\mathrm{d}t$
$\boldsymbol{J} = \int \boldsymbol{F}\,\mathrm{d}t = \Delta\boldsymbol{p}$	$\boldsymbol{J}_{\text{ang}} = \int \boldsymbol{\tau}\,\mathrm{d}t = \Delta\boldsymbol{L}$
$\boldsymbol{F} = \boldsymbol{0} \Rightarrow \boldsymbol{p} = \text{const}$	$\boldsymbol{\tau} = \boldsymbol{0} \Rightarrow \boldsymbol{L} = \text{const}$
$K = p^2/(2m)$	$K_{\text{rot}} = L^2/(2I)$

13.5 Conservation of Angular Momentum

Theorem 13.2: Conservation of Angular Momentum

If the net external torque about a point (or axis) is zero, the total angular momentum about that point is conserved:

$$\boxed{\boldsymbol{\tau}_{\text{ext}} = \boldsymbol{0} \quad\Longrightarrow\quad \boldsymbol{L}_{\text{total}} = \text{constant.}} \tag{13.7}$$

For rotation about a fixed axis: $I_i\omega_i = I_f\omega_f$.

This follows directly from Eq. (13.5): if $\boldsymbol{\tau}_{\text{ext}} = \boldsymbol{0}$, then $\mathrm{d}\boldsymbol{L}/\mathrm{d}t = \boldsymbol{0}$, so $\boldsymbol{L}$ is constant. Internal torques (no matter how large) cannot change the total angular momentum, just as internal forces cannot change the total linear momentum.

Consequences of $L = I\omega =$ const

When a system's moment of inertia changes while no external torque acts:

$$I_i\omega_i = I_f\omega_f \quad\Longrightarrow\quad \omega_f = \frac{I_i}{I_f}\omega_i.$$

If I decreases, ω increases. The kinetic energy also changes:

$$K = \frac{L^2}{2I}. \tag{13.8}$$

If I decreases while L is fixed, K *increases*.

Key Point 13.2: Energy Changes When I Changes

When I decreases and L is conserved, kinetic energy increases. The extra energy comes from *internal work*—the agent changing the configuration (e.g., a skater's muscles) does work against the centripetal acceleration. Conversely, when I increases, kinetic energy decreases and the agent absorbs energy.

Strategy 13.1: Angular Momentum Problems

1. **Choose the point/axis** about which to compute $\boldsymbol{L}$. Pick one where external torques vanish (e.g., a pivot, a frictionless axle, the force center for a central force, or a point where an unknown impulsive force acts).
2. **Compute L_i and L_f** about that point. Remember both the spin ($I\omega$) and orbital ($\boldsymbol{R} \times M\boldsymbol{v}$) contributions if relevant.
3. **Set** $L_i = L_f$ and solve for the unknown.
4. **Check energy**: if I changes, compute K_i and K_f separately: they will generally differ. Identify the source or sink of the energy difference (internal work, friction, gravitational PE, etc.).
5. **Ask**: is linear momentum also conserved? Often one is conserved and the other is not (e.g., a pivoted collision conserves L about the pivot but not $\boldsymbol{p}$, because the pivot exerts an external force).

Worked Examples

Example 13.2 (Ice skater). A skater with arms extended has $I_1 = 5.0\,\mathrm{kg\,m^2}$ and spins at $\omega_1 = 2$ rev/s. She pulls her arms in, reducing to $I_2 = 1.5\,\mathrm{kg\,m^2}$.

Solution. $\omega_2 = I_1\omega_1/I_2 = 5.0(2)/1.5 = 6.67$ rev/s.

$K_1 = \frac{1}{2}(5.0)(4\pi)^2 = 395\,\mathrm{J}$. $K_2 = \frac{1}{2}(1.5)(13.33\pi)^2 = 1317\,\mathrm{J}$.

The KE increased by 922 J—supplied by the skater's muscles doing work against the centrifugal tendency as the arms are pulled inward.

Example 13.3 (Ball on a string pulled through a hole). A point mass m moves in a circle of radius r_0 at speed v_0 on a frictionless table. The string is pulled through the hole, reducing the radius to $r_0/2$. Find the new speed and the work done.

Solution. The tension is radial ⇒ zero torque about the hole ⇒ L conserved:

$$mv_0r_0 = mv_f(r_0/2) \quad \Longrightarrow \quad v_f = 2v_0.$$

Work done by the tension (= change in KE):

$$W = \tfrac{1}{2}m(2v_0)^2 - \tfrac{1}{2}mv_0^2 = \tfrac{3}{2}mv_0^2.$$

The string tension does work here (unlike circular motion at fixed radius) because the mass moves radially inward, so $\boldsymbol{T} \cdot \mathrm{d}\boldsymbol{r} \neq 0$.

Example 13.4 (Neutron star). A star of mass M and radius R rotating with period T_0 collapses to a neutron star of radius $R_f = R/10^5$.

Solution. Model as a uniform sphere: $I = \frac{2}{5}MR^2$. No external torque during collapse:

$$\omega_f = \left(\frac{R}{R_f}\right)^2 \omega_i = 10^{10}\omega_i.$$

If $T_0 = 30$ days: $T_f = T_0/10^{10} \approx 0.26\,\text{ms}$—the star rotates thousands of times per second.

$K_f/K_i = I_i/I_f = 10^{10}$. The energy comes from gravitational PE released during the collapse.

Example 13.5 (Disk dropped on spinning disk). A disk (mass M, radius R) rotates freely at ω_0. A second identical disk, initially at rest, is dropped concentrically onto the first. Friction between the disks brings them to a common ω_f.

Solution. L conserved (friction is internal to the two-disk system):

$$I\omega_0 = 2I\omega_f \quad\Longrightarrow\quad \omega_f = \omega_0/2.$$

Fraction of KE lost: $1 - (2I)(\omega_0/2)^2/(I\omega_0^2) = 50\%$.

The lost energy was dissipated as heat by friction as the disks slid against each other. This is the rotational analogue of a perfectly inelastic collision.

Example 13.6 (Ball strikes rod—angular impulse). A uniform rod (M, L) hangs vertically from a pivot at one end. A ball (m, speed v_0) strikes the lower end and sticks.

Solution. Angular momentum about the pivot is conserved (the pivot exerts a force but zero torque about itself):

$$mv_0L = \left(\tfrac{1}{3}ML^2 + mL^2\right)\omega \quad\Longrightarrow\quad \omega = \frac{3mv_0}{(M+3m)L}.$$

Linear momentum is *not* conserved: the pivot delivers an external impulse. This illustrates a common pattern: choose the pivot as the reference point for L to exploit conservation, even though $\boldsymbol{p}$ is not conserved.

Example 13.7 (Child on merry-go-round). A child of mass m runs tangentially at speed v and jumps onto the rim of a stationary merry-go-round (uniform disk, M, R).

Solution. Before: $L_i = mvR$ (child's angular momentum about the center; this is the orbital contribution; there is no spin contribution). After: $L_f = (\frac{1}{2}MR^2 + mR^2)\omega$. Conserving L:

$$\omega = \frac{2mv}{(M+2m)R}.$$

Angular Momentum Conservation and Kepler's Second Law

For a particle moving under a central force $\boldsymbol{F} = F(r)\,\hat{\boldsymbol{r}}$, the torque about the force center vanishes: $\boldsymbol{\tau} = \boldsymbol{r} \times F(r)\,\hat{\boldsymbol{r}} = \boldsymbol{0}$. Therefore $\boldsymbol{L}$ is conserved.

In plane polar coordinates, $\boldsymbol{v} = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$, so $L = mr^2\dot{\theta}$. Since L is constant, the *areal velocity*—the rate at which the position vector sweeps out area—is:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \text{constant}. \tag{13.9}$$

This is **Kepler's second law**: the line from the Sun to a planet sweeps out equal areas in equal times. It is not a special property of gravity: it follows from angular momentum conservation for *any* central force; we shall discuss this in more detail in Chapter 16.

Problems

Problem 13.1 ⋆⋆⋆

A uniform disk of mass M and radius R is mounted on a frictionless axle and rotates freely at angular velocity ω_0. A person of mass m, initially standing at the edge of the disk, walks slowly inward to the center. Model the person as a point mass. (a) Find the total angular momentum of the system (disk + person) before the person walks. (b) Find the new angular velocity ω_1 after the person reaches the center. (c) Evaluate ω_1/ω_0 for the case $m = M$. (d) Compute the kinetic energy before and after. Show that KE increased, and explain where the extra energy came from.

Problem 13.2 ⋆⋆

A particle of mass m moves along a straight line with constant velocity $\boldsymbol{v}$. The line of motion passes at perpendicular distance d from a fixed point O. (a) Show that the angular momentum of the particle about O has magnitude $L = mvd$. (b) Show that L is constant in time, even though the particle is not moving in a circle. (c) Verify this by computing the torque about O and showing it is zero.

Problem 13.3 ⋆⋆⋆

A uniform rod of mass M and length L is pivoted at one end and hangs vertically at rest. A small ball of mass m moving horizontally at speed v_0 strikes the free (lower) end of the rod and sticks to it. (a) Explain why angular momentum about the pivot is conserved during the collision but linear momentum is not. (b) Find the angular velocity ω of the rod+ball system immediately after the collision. (c) Find the maximum angle θ_{max} through which the rod+ball swings upward after the collision, in terms of m, M, v_0, L, and g.

Problem 13.4 ⋆⋆⋆

A spinning top has moment of inertia $I = 4.0 \times 10^{-4}\,\text{kg}\,\text{m}^2$ and is initially spinning at 30 rev/s. A constant frictional torque of magnitude $\tau_f = 0.012\,\text{N}\,\text{m}$ acts on it. (a) Find the initial angular momentum. (b) Find the angular deceleration. (c) How long does it take to stop? (d) How many revolutions does it make before stopping?

Problem 13.5 ⋆⋆⋆⋆

A uniform thin rod of mass M and length L lies at rest on a frictionless horizontal surface. A point mass m moving at speed v_0 perpendicular to the rod strikes one end and sticks. (No pivot is present: the combined system translates and rotates freely.) (a) Find the velocity of the center of mass of the combined system after the collision. (b) Find the angular velocity ω about the center of mass after the collision. (c) What fraction of the initial kinetic energy is lost in the collision? (d) Evaluate all three quantities for the special case $m = M$.

Problem 13.6 ⋆⋆⋆

A child of mass $m = 30\,\text{kg}$ runs tangentially at speed $v = 4.0\,\text{m/s}$ and jumps onto the rim of a stationary merry-go-round, modeled as a uniform disk of mass $M = 100\,\text{kg}$ and radius $R = 2.0\,\text{m}$. (a) Find the angular velocity of the merry-go-round (with child) after the child lands. (b) What fraction of the child's initial kinetic energy is lost? (c) Where did the lost energy go?

Problem 13.7 ⋆⋆⋆⋆

A star of mass $M = 2M_\odot$ and radius $R = 7 \times 10^8$ m rotates with a period of $T = 30$ days. It collapses (conserving mass) to a neutron star of radius $R_f = 10\,\text{km}$. Model the star as a uniform solid sphere throughout. (a) Find the new rotation period. (b) By what factor does the rotational kinetic energy change? (c) Where does the enormous increase in rotational energy come from? (d) Is the final rotation speed relativistic? Estimate the equatorial surface speed and compare to c.

Problem 13.8 ⋆⋆⋆⋆

A uniform disk of mass M and radius R rotates freely about a vertical axis at angular velocity ω_0. A second identical disk, initially at rest, is dropped concentrically onto the first. Friction between the surfaces eventually brings them to a common angular velocity. (a) Find the final angular velocity ω_f. (b) Find the fraction of kinetic energy lost. (c) Explain where the lost energy went. (d) Is this the rotational analogue of an elastic or inelastic collision? Justify your answer.

Problem 13.9 ⋆⋆⋆⋆⋆

(*Kepler's second law.*) A particle of mass m moves under a central force $\boldsymbol{F} = F(r)\,\hat{\boldsymbol{r}}$. (a) Show that the torque about the force center is zero. (b) Deduce that $\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v}$ is conserved. (c) Working in plane polar coordinates with $\boldsymbol{v} = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$, show that $L = mr^2\dot{\theta}$. (d) The area swept out by the position vector in time $\mathrm{d}t$ is $\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta$. Deduce that $\mathrm{d}A/\mathrm{d}t = L/(2m) = \text{const}$, which is Kepler's second law.

Problem 13.10 ⋆⋆⋆⋆

A person of mass m stands at the rim of a stationary turntable (uniform disk, mass M, radius R, mounted on a frictionless axle). The person throws a ball of mass m_b horizontally at speed v_0 (relative to the ground) tangent to the rim. (a) What is the angular velocity of the turntable+person system after the throw? (b) Is kinetic energy conserved? If not, where did the extra energy come from? (c) If the person instead throws the ball radially outward, does the turntable rotate? Explain using angular momentum.

Chapter 14

Rigid Body Equilibrium

In Chapters 11–13 we studied objects in rotational *motion*—spinning, rolling, and colliding. In this chapter we study the opposite situation: objects that are *not* moving at all. A bridge, a building, a shelf bracket, a ladder against a wall—all must be designed so that every component is in **static equilibrium**.

The analysis of static equilibrium is among the oldest branches of physics (Archimedes studied levers and buoyancy in the 3rd century BCE), and it remains central to modern structural and mechanical engineering. The physics is deceptively simple (just two vector equations) but applying them to real structures requires careful attention to forces, their points of application, and the choice of pivot.

14.1 Conditions for Equilibrium

Theorem 14.1: Conditions for Static Equilibrium

A rigid body is in static equilibrium if and only if:

$$\boxed{\sum \boldsymbol{F}_{\text{ext}} = \mathbf{0} \qquad \text{(translational equilibrium)}} \tag{14.1}$$

$$\boxed{\sum \boldsymbol{\tau}_{\text{ext}} = \mathbf{0} \qquad \text{(rotational equilibrium)}} \tag{14.2}$$

Both conditions must hold simultaneously. The torques may be computed about *any* point: the result is the same (see below).

In two dimensions, these give three independent scalar equations: $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \tau_z = 0$. In three dimensions, there are six: three force components and three torque components.

Key Point 14.1: Both Conditions Are Necessary

Neither condition alone is sufficient. A body can have $\sum \boldsymbol{F} = \mathbf{0}$ but still spin (two equal and opposite forces forming a couple). Conversely, a body can have $\sum \boldsymbol{\tau} = \mathbf{0}$ about some point but still accelerate linearly (a single force through the pivot). *Both* must vanish for equilibrium.

Couples

A **couple** is a pair of forces that are equal in magnitude, opposite in direction, and applied at different points. A couple produces zero net force ($\sum \boldsymbol{F} = \mathbf{0}$) but a nonzero net torque. The torque of a couple is $\tau = Fd$, where d is the perpendicular distance between the lines of action of the two forces.

A key property: the torque of a couple is the *same about every point* (since $\sum \boldsymbol{F} = \mathbf{0}$, the proof above shows $\sum \boldsymbol{\tau}^P = \sum \boldsymbol{\tau}^O$ for all O and P). A couple produces pure rotation with no translation—turning a steering wheel, opening a jar lid, or the torque on a current loop in a magnetic field are all examples of couples.

Why Any Pivot Point Works

It might seem surprising that the torque condition $\sum \boldsymbol{\tau} = \mathbf{0}$ can be checked about *any* point, not just the center of mass or some "special" point. Here is why.

Suppose $\sum \boldsymbol{\tau}_O = \mathbf{0}$ about point O, and let P be any other point with $\boldsymbol{d} = \boldsymbol{r}_P - \boldsymbol{r}_O$. The torque of a force $\boldsymbol{F}_i$ applied at $\boldsymbol{r}_i$ about P is $\boldsymbol{\tau}_i^P = (\boldsymbol{r}_i - \boldsymbol{r}_P) \times \boldsymbol{F}_i = (\boldsymbol{r}_i - \boldsymbol{r}_O) \times \boldsymbol{F}_i - \boldsymbol{d} \times \boldsymbol{F}_i$. Summing:

$$\sum \boldsymbol{\tau}^P = \sum (\boldsymbol{r}_i - \boldsymbol{r}_O) \times \boldsymbol{F}_i - \boldsymbol{d} \times \sum \boldsymbol{F}_i = \sum \boldsymbol{\tau}^O - \boldsymbol{d} \times \sum \boldsymbol{F}_i.$$

If $\sum \boldsymbol{F} = \mathbf{0}$ (translational equilibrium), the second term vanishes, so $\sum \boldsymbol{\tau}^P = \sum \boldsymbol{\tau}^O$. Therefore: *if the forces balance and the torques vanish about one point, they vanish about every point.*

This freedom to choose *any* pivot is the single most powerful tool in equilibrium problems. A judicious choice places the pivot where unknown forces act, so those forces produce zero torque and drop out of the equation.

Center of Gravity

In writing the gravitational torque, we need the *point of application* of the weight. For a uniform gravitational field ($\boldsymbol{g}$ constant), the total gravitational torque about any point O is:

$$\boldsymbol{\tau}_g = \sum_i m_i \boldsymbol{r}_i \times \boldsymbol{g} = \left(\sum_i m_i \boldsymbol{r}_i \right) \times \boldsymbol{g} = M \boldsymbol{R}_{\text{cm}} \times \boldsymbol{g}.$$

This is the same as a single force $M\boldsymbol{g}$ acting at the center of mass. Therefore, in a uniform gravitational field, the **center of gravity** (the point where gravity "effectively acts") coincides with the center of mass.

In a non-uniform gravitational field (relevant for very tall structures or astrophysical bodies), the center of gravity can differ from the center of mass, but for all problems in this course they are identical.

14.2 Types of Supports and Their Constraints

Different types of mechanical supports constrain forces in different ways. Understanding these is essential for drawing correct free-body diagrams:

Support Type	**Unknown Forces**	**Example**
Frictionless surface/roller	1 (normal, $\perp$ to surface)	Ball on a floor
Rough surface	2 (normal + friction)	Ladder on rough floor
Pin/hinge (2D)	2 (H_x and H_y)	Door hinge
Cable/string	1 (tension, along the cable)	Hanging sign
Fixed/built-in wall (2D)	3 (H_x, H_y, and a moment)	Flagpole in a wall

A 2D equilibrium problem provides three equations ($F_x = 0$, $F_y = 0$, $\tau = 0$). If there are more than three unknown force components, the problem is **statically indeterminate**—additional information about the elasticity of the structure is needed.

14.3 Problem-Solving Strategy

Strategy 14.1: Static Equilibrium Problems

1. **Draw a free-body diagram** showing *all* external forces and their *points of application*. Include the weight, acting at the center of mass (= center of gravity).
2. **Choose coordinates** and write $\sum F_x = 0$ and $\sum F_y = 0$.
3. **Choose a pivot point** for the torque equation. The best choice is a point where unknown forces act (so they produce zero torque and are eliminated from $\sum \tau = 0$). You may use more than one pivot point if it simplifies the algebra.
4. **Compute torques** using $\tau = r_\perp F$ or $\tau = rF\sin\phi$. Be careful with signs: counterclockwise = positive.
5. **Solve** the resulting system of equations. In 2D, you have three equations for (up to) three unknowns.
6. **Check**: do the forces point in physically reasonable directions? Does $f_s \le \mu_s N$?

14.4 Worked Examples

Example 14.1 (Ladder against a wall). A uniform ladder of mass M and length L leans against a *frictionless* wall at angle θ from the floor. The floor has static friction coefficient μ_s. Find all forces and the minimum angle before the ladder slips.

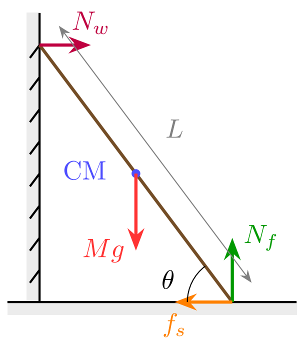


Figure 14.4.1: Free-body diagram for a uniform ladder against a frictionless wall.

Solution. Four forces act: Mg (down, at CM), N_f (up, at base), f_s (horizontal, at base), and N_w (horizontal, at top). The wall is frictionless, so the wall exerts only a horizontal normal force.

Torques about the base (eliminates N_f and f_s, which act at the pivot):

$$N_w L\sin\theta = Mg\frac{L}{2}\cos\theta \implies N_w = \frac{Mg}{2}\cot\theta.$$

Force equations: $\sum F_x = 0$: $f_s = N_w = \frac{Mg}{2}\cot\theta$. $\sum F_y = 0$: $N_f = Mg$.

Slip condition: The ladder doesn't slip as long as $f_s \le \mu_s N_f$:

$$\frac{Mg}{2}\cot\theta \le \mu_s Mg \implies \cot\theta \le 2\mu_s \implies \theta_{\min} = \arctan\left(\frac{1}{2\mu_s}\right).$$

For $\mu_s = 0.5$: $\theta_{\min} = \arctan(1) = 45°$. Steeper ladders are safer.

Example 14.2 (Sign on a beam). A uniform horizontal beam of mass $M = 20\,\text{kg}$ and length $L = 3.0\,\text{m}$ is attached to a wall by a hinge. A cable at angle $\alpha = 30°$ above horizontal supports the far end. A sign of mass $m = 10\,\text{kg}$ hangs from the far end. Find all forces.

Solution. Torques about the hinge (eliminates the unknown hinge force):

$$T \sin\alpha \cdot L = Mg \cdot \frac{L}{2} + mg \cdot L \quad \Longrightarrow \quad T = \frac{(M/2 + m)g}{\sin\alpha} = \frac{(10 + 10)(9.8)}{0.5} = 392\,\text{N}.$$

From $\sum F_x = 0$: $H_x = T\cos\alpha = 392 \cos 30° = 339\,\text{N}$.

From $\sum F_y = 0$: $H_y = (M + m)g - T\sin\alpha = 294 - 196 = 98\,\text{N}$.

Hinge force: $H = \sqrt{339^2 + 98^2} = 353\,\text{N}$ at $\arctan(98/339) = 16.1°$ above horizontal.

Example 14.3 (Person on a plank). A uniform plank of mass $M = 40\,\text{kg}$ and length $L = 5.0\,\text{m}$ is supported at both ends. A person of mass $m = 70\,\text{kg}$ stands 2.0 m from the left support. Find the support forces.

Solution. Torques about the left support:

$$N_R L = Mg\frac{L}{2} + mg \cdot d = 40(9.8)(2.5) + 70(9.8)(2.0) = 980 + 1372 = 2352\,\text{N m}.$$

$$N_R = 2352/5.0 = 470\,\text{N}.$$

From $\sum F_y = 0$: $N_L = (M + m)g - N_R = 110(9.8) - 470 = 608\,\text{N}$.

Check: the person is closer to the left support, so $N_L > N_R$. ✓

Example 14.4 (Cable sag). A traffic light of mass $m = 20\,\text{kg}$ hangs from the midpoint of a cable strung between two poles separated by $d = 12\,\text{m}$. The cable sags by $h = 0.50\,\text{m}$. Find the tension.

Solution. The half-cable makes angle α with horizontal: $\tan\alpha = h/(d/2) = 0.50/6.0 = 0.0833$, so $\alpha = 4.76°$. At the midpoint, the vertical components of the two half-cable tensions must support the weight:

$$2T\sin\alpha = mg \quad \Longrightarrow \quad T = \frac{mg}{2\sin\alpha} = \frac{20(9.8)}{2(0.0831)} = 1180\,\text{N}.$$

This is 6 times the weight of the traffic light! As $h \to 0$ (less sag), $\alpha \to 0$ and $T \to \infty$: *a perfectly horizontal cable would require infinite tension*, which is why real cables always sag.

Example 14.5 (Tipping vs. sliding). A uniform cube of side a sits on a rough incline. At what angle does it tip versus slide?

Solution. Sliding occurs when $mg\sin\theta > \mu_s mg\cos\theta$, i.e., $\tan\theta > \mu_s$, so $\theta_{\text{slide}} = \arctan\mu_s$.

Tipping occurs when the center of mass passes over the downhill edge. For a cube, the CM is at the geometric center, distance $a/2$ above the base and $a/2$ from each edge. The cube tips when the vertical line through the CM falls outside the base, which happens at $\theta_{\text{tip}} = 45°$ (since the CM-to-edge and CM-to-base distances are equal for a cube).

The cube tips before sliding if $\theta_{\text{tip}} < \theta_{\text{slide}}$, i.e., $45° < \arctan\mu_s$, i.e., $\mu_s > 1$. For $\mu_s < 1$: slides first. For $\mu_s > 1$: tips first.

14.5 Stability of Equilibrium

Not all equilibria are created equal. An equilibrium is:

- **Stable** if a small displacement produces a restoring force/torque that returns the object to equilibrium (e.g., a ball at the bottom of a bowl).

- **Unstable** if a small displacement produces a force/torque that drives the object further from equilibrium (e.g., a ball balanced on top of a hill).
- **Neutral** if a small displacement produces no net force/torque (e.g., a ball on a flat surface).

For a rigid body supported from below, the criterion involves the **base of support**—the convex hull of all contact points. The equilibrium is stable as long as the vertical line through the center of gravity falls within the base of support. If the CG's vertical projection falls outside the base, the gravitational torque tips the object over.

This principle explains many everyday observations: a wider stance is more stable because it enlarges the base of support; loaded trucks are prone to tipping on curves because the load raises the CG; and a person carrying a heavy suitcase in one hand leans the opposite way to keep their combined CG above their feet.

Example 14.6 (Stability of a loaded shelf). A uniform bookshelf of mass $M = 15\,\text{kg}$, height $H = 1.8\,\text{m}$, and depth $D = 0.30\,\text{m}$ stands on the floor. Books of total mass $m = 40\,\text{kg}$ are placed uniformly on the top shelf (at height H). Does the bookshelf tip?

Solution. The base of support extends from the front to the back of the shelf: width $D = 0.30$ m. The combined CG is at horizontal position $D/2$ (center of shelf) and height:

$$y_{\text{cg}} = \frac{M(H/2) + m(H)}{M + m} = \frac{15(0.9) + 40(1.8)}{55} = \frac{13.5 + 72}{55} = 1.55\,\text{m}.$$

The CG is directly above the center of the base, so in the absence of any horizontal force, the shelf does not tip. However, if the shelf is tilted by even a small angle θ, the CG shifts horizontally by approximately $y_{\text{cg}} \sin\theta$. Tipping occurs when this shift exceeds $D/2$:

$$y_{\text{cg}} \sin\theta_{\text{tip}} = D/2 \quad \Longrightarrow \quad \theta_{\text{tip}} = \arcsin\left(\frac{D}{2y_{\text{cg}}}\right) = \arcsin\left(\frac{0.15}{1.55}\right) \approx 5.6^\circ.$$

A very small disturbance can tip a tall, top-heavy shelf; this is why bookshelves should be anchored to the wall.

14.6 Elasticity: Stress, Strain, and Elastic Moduli

In the preceding sections we treated objects as perfectly *rigid*—unable to deform under any load. Real materials, of course, deform when forces act on them: cables stretch, beams bend, and columns compress. The study of how materials respond to forces is called **elasticity**.

Understanding elasticity answers the question that static equilibrium alone cannot: *will the structure survive?* Knowing the forces in a beam or cable is only the first step; we must also ask whether those forces exceed the material's capacity.

Stress and Strain

Definition 14.1: Stress and Strain

Stress is the force per unit area acting on a material:

$$\sigma = \frac{F}{A} \qquad [\text{Pa} = \text{N/m}^2]. \tag{14.3}$$

Strain is the dimensionless fractional deformation:

$$\varepsilon = \frac{\Delta L}{L_0} \qquad \text{(tensile/compressive)}. \tag{14.4}$$

Stress is the *cause* (applied force per unit area); strain is the *effect* (resulting deformation). Both are intensive quantities: they do not depend on the size or shape of the object, only on the material.

There are three basic types of stress and corresponding strain:

- **Tensile stress**: a pulling (stretching) force per unit cross-sectional area. The object elongates.
- **Compressive stress**: a pushing (squeezing) force per unit cross-sectional area. The object shortens.
- **Shear stress**: a force tangent to a surface, per unit area. The object distorts laterally (like a deck of cards sliding).

Elastic Moduli: Hooke's Law for Materials

For small deformations, most materials obey a linear relationship between stress and strain: the material analogue of Hooke's law $F = kx$:

$$\boxed{\sigma = E\,\varepsilon \qquad \Longleftrightarrow \qquad \frac{F}{A} = E\,\frac{\Delta L}{L_0},} \tag{14.5}$$

where E is the **Young's modulus** (or elastic modulus) of the material. Young's modulus has units of pressure (Pa) and characterizes the stiffness of a material: a large E means the material is stiff (hard to stretch), a small E means it is compliant (easy to stretch).

Key Point 14.2: Connection to Springs

A rod of length L_0, cross-sectional area A, and Young's modulus E behaves as a spring with spring constant:

$$k = \frac{EA}{L_0}. \tag{14.6}$$

This follows directly from rearranging $F = (EA/L_0)\,\Delta L = k\,\Delta L$. A shorter, thicker rod is stiffer; a longer, thinner rod is more compliant.

For **shear deformation**, the analogous relationship is $\tau_{\text{shear}} = G\gamma$, where G is the **shear modulus** and $\gamma = \Delta x/h$ is the shear strain (lateral displacement divided by height).

For **uniform compression** (pressure applied on all sides), the relationship is $\Delta P = -B\,(\Delta V/V_0)$, where B is the **bulk modulus**.

Material	**Young's E (GPa)**	**Shear G (GPa)**	**Ultimate strength (MPa)**
Steel	200	80	400–800
Aluminum	70	26	90–570
Copper	120	45	220
Bone (compact)	15–20	3	100–120 (tension)
Concrete	30	—	2–5 (tension), 20–40 (compression)
Rubber	0.01–0.1	~ 0.001	15–20
Nylon	2–5	—	70–85

Beyond the Linear Regime: Breaking

The linear stress-strain relationship holds only up to the **proportional limit**. Beyond this, the material yields (deforms permanently) and eventually fractures at the **ultimate tensile strength** σ_{UTS}. The maximum force a rod can sustain before breaking is:

$$F_{\max} = \sigma_{\text{UTS}} \cdot A. \tag{14.7}$$

Example 14.7 (Steel cable). A steel cable of diameter $d = 1.0\,\text{cm}$ and length $L_0 = 10\,\text{m}$ supports a load of $F = 5000\,\text{N}$. Find (a) the stress, (b) the strain, and (c) the elongation.

Solution. (a) $A = \pi(d/2)^2 = \pi(0.005)^2 = 7.85 \times 10^{-5}\,\text{m}^2$. $\sigma = F/A = 5000/7.85 \times 10^{-5} = 6.37 \times 10^7\,\text{Pa} = 63.7\,\text{MPa}$.

(b) $\varepsilon = \sigma/E = 6.37 \times 10^7/(200 \times 10^9) = 3.18 \times 10^{-4} \approx 0.032\%$.

(c) $\Delta L = \varepsilon L_0 = 3.18 \times 10^{-4} \times 10 = 3.2\,\text{mm}$.

The cable stretches by only 3 mm under a half-ton load—steel is extremely stiff. The stress (64 MPa) is well below the ultimate strength (∼500 MPa), so the cable is in no danger of breaking.

Example 14.8 (Will the bone break?). The femur (thigh bone) has a cross-sectional area of approximately $A \approx 6.0\,\text{cm}^2$ and an ultimate compressive strength of about 170 MPa. What is the maximum compressive force it can withstand?

Solution. $F_{\max} = \sigma_{\text{UTS}} \times A = 170 \times 10^6 \times 6.0 \times 10^{-4} = 102\,000\,\text{N} \approx 100\,\text{kN}$. This is roughly 150 times body weight for a 70 kg person—bones are remarkably strong in compression. However, bones are much weaker in *bending* and *torsion*, which is why fractures typically occur from impacts that produce bending or twisting forces rather than pure compression.

Problems

Problem 14.1 ⋆⋆⋆⋆

A uniform beam of mass M and length L is attached to a wall by a hinge at its left end. A cable at angle θ above horizontal is attached to the far (right) end. A block of mass m hangs from the far end. (a) Taking torques about the hinge, derive the cable tension T. (b) Find the horizontal and vertical components of the hinge force. (c) For what angle θ is the cable tension minimized? What is $T_{\min}$? (d) Compute all forces for $M = 20\,\text{kg}$, $m = 50\,\text{kg}$, $\theta = 30°$, $g = 10\,\text{m/s}^2$.

Problem 14.2 ⋆⋆

A uniform plank of mass M and length L rests horizontally on two supports, one at each end. A person of mass m stands at distance d from the left support. (a) Find the normal force at each support. (b) Verify that the forces sum to $(M + m)g$. (c) Where should the person stand so that both supports bear equal load?

Problem 14.3 ⋆⋆⋆

A uniform ladder of mass M and length L leans at angle θ against a frictionless wall. The floor has static friction coefficient μ_s. (a) Draw a free-body diagram and find all four forces acting on the ladder. (b) Find the minimum angle $\theta_{\min}$ at which the ladder can stand without slipping. (c) A person of mass m climbs to a fraction f of the way up the ladder. How does $\theta_{\min}$ change?

Problem 14.4 ⋆⋆⋆

A crane arm (uniform beam of mass M and length L) is attached at a pivot at its base and held at angle θ above horizontal by a horizontal cable attached at the midpoint $L/2$. A load of mass m hangs from the tip. (a) Find the cable tension. (b) Find the magnitude and direction of the pivot force.

Problem 14.5 ⋆⋆⋆⋆

A uniform cube of side a and mass M sits on a rough inclined plane of angle θ with static friction coefficient μ_s. (a) Find the angle θ_{slide} at which the cube begins to slide. (b) Find the angle θ_{tip} at which the cube tips over. (c) For what values of μ_s does tipping occur before sliding?

Problem 14.6 ⋆⋆⋆

A traffic light of mass m hangs from the midpoint of a cable strung between two poles separated by distance d. The cable sags by amount h at the center ($h \ll d$). (a) Find the tension in the cable. (b) Evaluate for $m = 20\,\text{kg}$, $d = 12\,\text{m}$, $h = 0.5\,\text{m}$. (c) Explain physically why the cable can never be perfectly horizontal ($h = 0$).

Problem 14.7 ⋆⋆⋆

A person of mass M does a push-up. Model the body as a uniform rigid plank of length L pivoted at the toes, with the hands placed at distance d from the shoulder (head) end. (a) Find the force each hand must exert. (b) Evaluate the fraction of body weight supported by the hands for $d \ll L$. (c) Why do push-ups become harder when the hands are placed farther from the shoulders?

Problem 14.8 ⋆⋆⋆⋆

A uniform disk of mass M and radius R rests on a rough incline of angle θ. It is held in place by a horizontal string attached to its highest point. (a) Draw a free-body diagram showing all forces and their points of application. (b) Taking torques about the contact point, find the string tension. (c) Find the normal force and the friction force at the contact point.

Problem 14.9 ⋆⋆⋆

A seesaw is modeled as a uniform plank of mass M and length L, supported at its midpoint. Child A (mass m_A) sits at distance d_A from the center, and child B (mass m_B) sits at distance d_B on the other side. (a) For equilibrium, find the relation between $m_A d_A$ and $m_B d_B$. (b) Find the force exerted on the pivot. (c) If $m_A = 25\,\mathrm{kg}$, $m_B = 40\,\mathrm{kg}$, and $d_A = 2.0\,\mathrm{m}$, find d_B.

Problem 14.10 ⋆⋆⋆⋆⋆

A non-uniform beam of length L and total mass M has its center of mass at distance $L/3$ from the left end. It is supported by a pin at the left end and a cable at the right end making angle α with the beam. A block of mass m hangs from the beam at a variable position x measured from the left end. (a) Find the cable tension $T(x)$ as a function of x. (b) Find the pin-force components as functions of x. (c) For what position x is the cable tension minimized? Maximized? (d) At what x does the horizontal pin force vanish?

Problem 14.11 ⋆⋆⋆

A steel wire of diameter $d = 2.0\,\mathrm{mm}$ and length $L_0 = 3.0\,\mathrm{m}$ is used to hang a chandelier of mass $m = 25\,\mathrm{kg}$. (a) Find the stress in the wire. (b) Find the strain and the elongation. Use $E_{\mathrm{steel}} = 200\,\mathrm{GPa}$. (c) What is the maximum mass the wire can support before reaching the ultimate tensile strength of $\sigma_{\mathrm{UTS}} = 500\,\mathrm{MPa}$?

Problem 14.12 ⋆⋆⋆⋆

Two vertical wires, one of steel ($E_s = 200\,\mathrm{GPa}$, A_s) and one of aluminum ($E_a = 70\,\mathrm{GPa}$, A_a), support a rigid horizontal bar of mass M at its two ends. Both wires have the same natural length L_0. (a) If $A_s = A_a = A$, find the fraction of the weight supported by each wire. (b) Find the ratio A_s/A_a such that both wires stretch by the same amount.

Chapter 15

Gravitation

Gravitation is the oldest known fundamental force and the first to be described by a precise mathematical law. Newton's insight—that the same force that makes an apple fall also holds the Moon in its orbit—unified terrestrial and celestial mechanics in a single framework. As Newton wrote in the *Principia* (1687): the force of gravity acts on all bodies in proportion to their mass, and diminishes as the inverse square of the distance.

In this chapter we develop Newton's law of universal gravitation, the gravitational field, gravitational potential energy, the shell theorem, and escape velocity. Orbital mechanics and Kepler's laws are treated in Chapter 16.

15.1 Newton's Law of Universal Gravitation

Theorem 15.1: Universal Gravitation

Every particle of mass m_1 attracts every other particle of mass m_2 with a force directed along the line joining them:

$$\boxed{F = \frac{Gm_1m_2}{r^2}, \qquad G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}.} \tag{15.1}$$

In vector form (force on m_2 due to m_1 at the origin):

$$\boldsymbol{F}_{12} = -\frac{Gm_1m_2}{r^2}\hat{\boldsymbol{r}}, \tag{15.2}$$

where $\hat{\boldsymbol{r}}$ points from m_1 to m_2 and the negative sign indicates attraction.

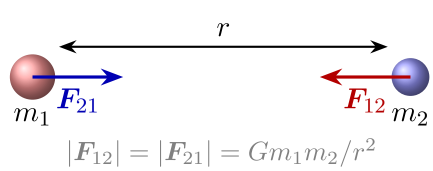


Figure 15.1.1: Newton's law of universal gravitation: two masses attract each other with equal and opposite forces of magnitude Gm_1m_2/r^2.

Several features of the law deserve emphasis:

- **Universal**: it applies to *all* masses (planets, stars, apples, atoms) without exception.
- **Inverse-square**: the force drops as $1/r^2$. Double the distance and the force drops to a quarter.
- **Always attractive**: unlike electric forces, gravity is never repulsive.

- **Obeys Newton's third law**: $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$. The Earth pulls on the Moon with the same force that the Moon pulls on the Earth.
- **Extremely weak**: G is tiny. The gravitational attraction between two 1 kg masses 1 m apart is only 6.7×10^{-11} N—about the weight of a few bacteria. Yet gravity dominates the cosmos because it is always attractive and has infinite range.

Measuring G: The Cavendish Experiment

The constant G was first measured by Henry Cavendish in 1798, using a torsion balance: two small lead spheres attached to a lightweight bar are attracted by two large lead spheres, twisting a thin wire. The deflection angle measures the gravitational force, from which G is extracted. Since G and $g_s = GM_\oplus/R_\oplus^2$ are both known, this experiment effectively "weighs the Earth": $M_\oplus = g_s R_\oplus^2/G \approx 5.97 \times 10^{24}$ kg.

Surface Gravity and the Connection to mg

At the surface of a planet of mass M_p and radius R, the gravitational field strength is:

$$g_s = \frac{GM_p}{R^2}. \tag{15.3}$$

This is the g in $F = mg$. For Earth: $g_s = 9.80\,\mathrm{m/s^2}$. The "flat-Earth" approximation $F = mg$ is valid whenever the object stays close to the surface ($h \ll R$) so that g is approximately constant.

Example 15.1 (Surface gravity on another planet). Venus has mass $0.815M_\oplus$ and radius $0.949R_\oplus$. Find the surface gravity and compare to Earth.

Solution. $g_V = G(0.815M_\oplus)/(0.949R_\oplus)^2 = (0.815/0.9006)g_\oplus = 0.905g_\oplus = 8.87\,\mathrm{m/s^2}$. A 75 N rock on Earth weighs $(75/9.8)(8.87) = 67.9$ N on Venus, about 91% of its Earth weight.

Superposition

Gravity obeys the **principle of superposition**: the gravitational force on a particle due to several other masses is the vector sum of the individual forces:

$$\boldsymbol{F}_{\text{net}} = \sum_i \boldsymbol{F}_i = -\sum_i \frac{Gm\,m_i}{r_i^2}\,\hat{\boldsymbol{r}}_i.$$

For continuous mass distributions, the sum becomes an integral.

Example 15.2 (Three masses in a triangle). Three identical masses m are at the vertices of an equilateral triangle of side a. Find the net force on one of them.

Solution. Each neighbor exerts $F = Gm^2/a^2$ on the third mass. By symmetry, the net force points toward the centroid. The component from each neighbor toward the centroid is $F\cos 30^\circ = F\sqrt{3}/2$. The two contributions add: $F_{\text{net}} = 2F\cos 30^\circ = Gm^2\sqrt{3}/a^2$.

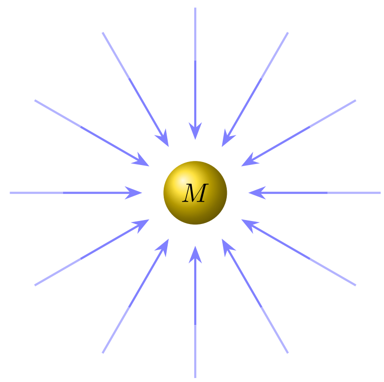
15.2 The Gravitational Field

Definition 15.1: Gravitational Field

The gravitational field at a point in space is the gravitational force per unit mass experienced by a test mass at that point:

$$\boxed{\boldsymbol{g}(\boldsymbol{r}) = -\frac{GM}{r^2}\hat{\boldsymbol{r}}.} \tag{15.4}$$

For multiple masses, $\boldsymbol{g}$ is found by superposition: $\boldsymbol{g} = \sum_i \boldsymbol{g}_i$.



Gravitational field lines point radially inward

Figure 15.2.1: Gravitational field lines around a point mass M. The field points radially inward and weakens as $1/r^2$.

The field concept is useful because $\boldsymbol{g}(\boldsymbol{r})$ is a property of *space* (created by the source mass M), independent of whatever test mass we place there. The force on any mass m placed in the field is simply $\boldsymbol{F} = m\boldsymbol{g}$.

Variation of g with Altitude

At height h above the surface of a planet of radius R:

$$g(h) = \frac{GM_p}{(R+h)^2} = g_s\left(\frac{R}{R+h}\right)^2. \tag{15.5}$$

For $h \ll R$: $g(h) \approx g_s(1 - 2h/R)$—gravity decreases approximately linearly with altitude near the surface.

Example 15.3 (Altitude for half-gravity). At what altitude above the Earth is g reduced to half its surface value?

Solution. $g(R+h) = g_s/2$: $(R+h)^2 = 2R^2$, so $h = (\sqrt{2}-1)R \approx 0.414R \approx 2640\,\text{km}$.

Note this is *not* $R/2$—the inverse-square law means you must go surprisingly far to significantly reduce gravity. At the altitude of the ISS ($h \approx 400\,\text{km}$), g is still about 89% of its surface value. Astronauts are “weightless” not because gravity is absent, but because they (and the station) are in free fall.

Key Point 15.1: Weightlessness in Orbit

An astronaut in orbit experiences *apparent* weightlessness, not because gravity is weak (it's nearly as strong as at the surface), but because the astronaut and the spacecraft are both in free fall. There is no normal force, so the apparent weight (what a scale reads) is zero. This is the same physics as the momentary weightlessness at the top of a roller-coaster hill.

15.3 Gravitational Potential Energy

Theorem 15.2: Gravitational Potential Energy

The gravitational potential energy of a mass m at distance r from a mass M, with $U \to 0$ as $r \to \infty$, is:

$$\boxed{U(r) = -\frac{GMm}{r}.} \tag{15.6}$$

Derivation. The work done by gravity as m moves from ∞ to r:

$$U(r) = -\int_\infty^r \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}' = -\int_\infty^r \left(-\frac{GMm}{r'^2}\right) \mathrm{d}r' = -\frac{GMm}{r}.$$

The negative sign means the system has *less* energy when the masses are close—bound states have $E < 0$. The zero of potential energy is at infinity, which is the natural choice for gravitational problems (as opposed to the arbitrary "ground level" of $U = mgh$).

Connection to the Flat-Earth Formula

Near the surface ($r = R + h$, $h \ll R$):

$$U = -\frac{GMm}{R+h} = -\frac{GMm}{R} \cdot \frac{1}{1 + h/R} \approx -\frac{GMm}{R} + \frac{GMm}{R^2} h = \text{const} + mg_s h.$$

This recovers $U = mgh$ (up to an irrelevant constant), confirming that the flat-Earth approximation is the small-h limit of the full gravitational PE.

Gravitational Energy Diagrams

The total mechanical energy of a mass m moving in the gravitational field of M is:

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}. \tag{15.7}$$

Since $K \geq 0$, the mass is **bound** (r stays finite) when $E < 0$ and **unbound** (escapes to infinity) when $E \geq 0$. The boundary case $E = 0$ corresponds to escape velocity: the mass reaches infinity with zero speed.

Example 15.4 (Launch to a given height). An object is launched vertically at speed v_0 from the surface of a planet of mass M_p and radius R. Find the maximum height h.

Solution. Energy conservation ($K_f = 0$ at the top):

$$\frac{1}{2}mv_0^2 - \frac{GM_p m}{R} = -\frac{GM_p m}{R+h}.$$

Solving for h:

$$h = \frac{v_0^2 R}{2g_s R - v_0^2} = \frac{R}{(v_e/v_0)^2 - 1}.$$

For $v_0 \ll v_e$: $h \approx v_0^2/(2g_s)$ (flat-Earth limit). As $v_0 \to v_e$: $h \to \infty$ (escape).

For Earth with $v_0 = 5.0\,\text{km/s}$: $h = (25 \times 10^6)(6.37 \times 10^6)/((2)(9.8)(6.37 \times 10^6) - 25 \times 10^6) \approx 1600\,\text{km}$. The flat-Earth formula gives $h = v_0^2/(2g) = 1276\,\text{km}$—an underestimate of about 20%.

15.4 Escape Velocity

Definition 15.2: Escape Velocity

The minimum launch speed needed for an object to escape to infinity (with zero residual speed):

$$\boxed{v_e = \sqrt{\frac{2GM_p}{R}} = \sqrt{2g_s R}.} \tag{15.8}$$

Derivation. Energy conservation with $K_\infty = 0$, $U_\infty = 0$:

$$\frac{1}{2}mv_e^2 - \frac{GM_p m}{R} = 0 \quad \Longrightarrow \quad v_e = \sqrt{\frac{2GM_p}{R}}.$$

For Earth: $v_e \approx 11.2\,\text{km/s}$. For the Moon: $v_e \approx 2.4\,\text{km/s}$.

Key Point 15.2: Escape Velocity Is Independent of Direction

The escape velocity depends only on speed, not direction, an object launched horizontally at v_e escapes just as surely as one launched vertically (assuming no atmosphere to provide drag). This is because kinetic energy $\frac{1}{2}mv^2$ depends only on speed. Of course, a horizontal launch skims close to the surface and may collide with terrain, but the energy argument is the same.

Example 15.5 (Escape from the Moon). $v_e = \sqrt{2GM_M/R_M} = \sqrt{2(6.674 \times 10^{-11})(7.35 \times 10^{22})/(1.74 \times 10^{6}$ 2.38 km/s. This is about 1/5 of Earth's escape velocity, which is why the Moon cannot retain a significant atmosphere: the thermal velocities of gas molecules exceed v_e.

15.5 The Shell Theorem

Theorem 15.3: Newton's Shell Theorem

1. A uniform spherical shell attracts an external particle as if all the shell's mass were concentrated at its center.
2. A uniform spherical shell exerts *zero* net gravitational force on a particle anywhere inside it.

This remarkable result means: (a) for an external object, a sphere acts as a point mass at its center; (b) for an internal object, only the mass at smaller radii contributes.

Proof of the Shell Theorem

Consider a thin uniform spherical shell of mass M_s and radius R. A test mass m sits at distance d from the center. Divide the shell into thin rings perpendicular to the line from the center to m, where each ring subtends angle $\mathrm{d}\psi$ at the center. The mass of a ring at angle ψ is $\mathrm{d}M = \frac{M_s}{2}\sin\psi\,\mathrm{d}\psi$. Every point on the ring is at distance s from m, where by the law of cosines: $s^2 = R^2 + d^2 - 2Rd\cos\psi$. Changing the integration variable from ψ to s:

$$F = -\frac{GmM_s}{4Rd^2}\int_{s_{\min}}^{s_{\max}}\left(1 + \frac{d^2 - R^2}{s^2}\right)\mathrm{d}s.$$

External case ($d > R$): limits $s_{\min} = d - R$, $s_{\max} = d + R$. The integral evaluates to $F = -GmM_s/d^2$—exactly a point mass at the center.

Internal case ($d < R$): limits $s_{\min} = R - d$, $s_{\max} = R + d$. The integral evaluates to zero: $F = 0$.

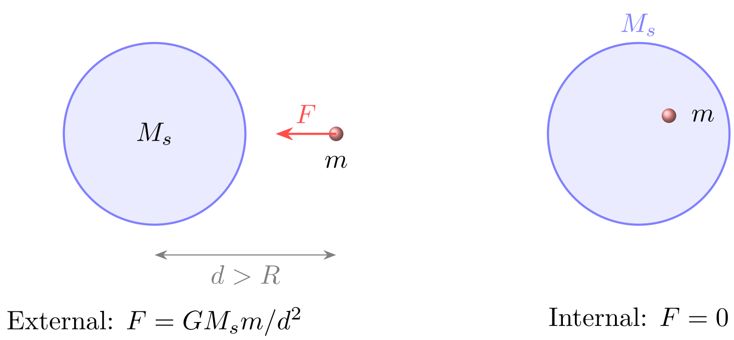


Figure 15.5.1: Newton's shell theorem. *Left:* external mass feels the shell as a point mass at the center. *Right:* internal mass feels zero net force.

Gravity Inside a Uniform Sphere

At radius $r \le R$ inside a uniform sphere of mass M_p, the shell theorem tells us that only the mass enclosed within radius r contributes. The enclosed mass is $M(r) = M_p(r/R)^3$ (volume scales as r^3). Therefore:

$$F(r) = \frac{GM(r)m}{r^2} = \frac{GM_pm}{R^3}r = \frac{mg_s}{R}r. \tag{15.9}$$

This is a **linear restoring force**—exactly the form $F = -k_{\text{eff}}r$ with $k_{\text{eff}} = mg_s/R$—meaning an object in a tunnel through a uniform planet executes **simple harmonic motion** with period:

$$T = 2\pi\sqrt{\frac{R}{g_s}}. \tag{15.10}$$

Remarkably, this equals the orbital period of a satellite skimming the surface. For Earth: $T \approx 84\,\text{min}$—the same whether you orbit the planet or tunnel through it. (Simple harmonic motion is developed in detail in Chapter 18.)

Example 15.6 (Tunnel through the Earth). An object is dropped into a frictionless tunnel through the center of the Earth. Find the speed at the center and the time to reach the other side.

Solution. By energy conservation, $\frac{1}{2}mv_c^2 = \frac{1}{2}mg_sR$ (the PE difference from surface to center for the linear-force model), giving $v_c = \sqrt{g_sR} = \sqrt{9.8 \times 6.37 \times 10^6} \approx 7.9\,\text{km/s}$.

The transit time is half a period: $t = T/2 = \pi\sqrt{R/g_s} \approx 42\,\text{min}$. This is the same regardless of whether the tunnel passes through the center or along any chord, a remarkable consequence of the linear restoring force (all chords give the same period, just as all amplitudes give the same period for SHM).

15.6 The Gravitational Potential*

(This section uses the gradient from multivariable calculus and may be treated as optional.)

The gravitational potential is the potential energy per unit mass:

$$\Phi(\boldsymbol{r}) = -\frac{GM}{r}, \qquad U = m\Phi, \qquad \boldsymbol{g} = -\nabla\Phi. \tag{15.11}$$

Poisson's and Laplace's Equations

The potential satisfies the fundamental field equation:

$$\nabla^2\Phi = 4\pi G\rho, \tag{15.12}$$

where $\rho(\boldsymbol{r})$ is the local mass density. This is **Poisson's equation**. In empty space ($\rho = 0$), it reduces to **Laplace's equation**: $\nabla^2\Phi = 0$.

A consequence of Laplace's equation is the **mean value property**: Φ at any point in empty space equals its average over any surrounding sphere. This immediately implies Result 2 of the shell theorem (no force inside a hollow shell), because if Φ is harmonic inside an empty region, it has no local extrema, so there is no direction in which $\boldsymbol{g}$ could point.

Potential Inside a Uniform Sphere

For a uniform sphere of density ρ_0, mass M_p, and radius R, solving Poisson's equation in spherical coordinates gives:

$$\Phi(r) = -\frac{GM_p}{2R}\left(3 - \frac{r^2}{R^2}\right), \qquad r \le R. \tag{15.13}$$

The field is $g(r) = -\mathrm{d}\Phi/\mathrm{d}r = (GM_p/R^3)r$, confirming the linear restoring force. The potential at the center is $\Phi(0) = -3GM_p/(2R) = \frac{3}{2}\Phi_{\text{surface}}$—the center is 50% deeper in the potential well than the surface.

Problems

Problem 15.1 ⋆

Find the gravitational force between the Earth ($M_E = 5.97 \times 10^{24}\,\text{kg}$) and the Moon ($M_M = 7.35 \times 10^{22}\,\text{kg}$) separated by $r = 3.84 \times 10^{8}\,\text{m}$. Express the result in newtons and compare it to the weight of a familiar object.

Problem 15.2 ⋆⋆

Venus has 81.5% of Earth's mass and 94.9% of Earth's radius. (a) Compute g_V. (b) What does a rock that weighs 75 N on Earth weigh on Venus?

Problem 15.3 ⋆⋆

At what altitude above the Earth's surface is g reduced to half its surface value? Compare this to the altitude of the International Space Station ($h \approx 400\,\text{km}$) and explain why astronauts aboard the ISS are "weightless" even though gravity is nearly as strong as at the surface.

Problem 15.4 ⋆⋆⋆

An object is launched vertically from the surface of a planet of mass M_p and radius R at speed $v_0 < v_e$. (a) Using energy conservation, derive the maximum height h in terms of v_0, g_s, and R. (b) Show that $h \to v_0^2/(2g_s)$ in the limit $h \ll R$ (flat-Earth approximation). (c) Show that $v_0 \to v_e$ as $h \to \infty$. (d) Compute h for Earth with $v_0 = 5.0\,\text{km/s}$ and compare to the flat-Earth result.

Problem 15.5 ⋆⋆⋆⋆

A uniform spherical planet of mass M_p and radius R has a frictionless tunnel drilled straight through its center. (a) Show that the gravitational force on a mass m at distance r from the center is $F = -mg_s r/R$. (b) Show that the motion is simple harmonic and find the period. (c) Show that this period equals the orbital period of a satellite skimming the surface. (d) Find the speed of the object as it passes through the center. (e) Compute the period and center speed for Earth.

Problem 15.6 ⋆⋆⋆

Three identical masses m are at the vertices of an equilateral triangle of side a. Find the magnitude and direction of the net gravitational force on one of them.

Problem 15.7 ⋆⋆⋆

A spherical planet has density $\rho(r) = \rho_0(1 - r/R)$ for $r \le R$. (a) Find the total mass M. (b) Find $g(r)$ for $r \le R$. (c) At what radius r is $g(r)$ maximized? Interpret physically.

Problem 15.8 ⋆⋆⋆⋆

Derive the gravitational self-energy of a uniform sphere of mass M and radius R: $U_{\text{self}} = -3GM^2/(5R)$. (*Hint:* build the sphere shell by shell. When the sphere has been built out to radius r, the energy needed to add the next thin shell of mass $\mathrm{d}m$ is $\mathrm{d}U = -GM(r)\,\mathrm{d}m/r$.)

Problem 15.9 ⋆⋆⋆⋆

A satellite in a circular orbit of radius r_0 about a planet of mass M is given a small radial velocity perturbation δv. Using the effective potential $V_{\text{eff}}(r) = -GMm/r + L^2/(2mr^2)$, show that the satellite oscillates radially about r_0 with the same frequency as the orbital frequency. (*Hint:* Taylor-expand V_{eff} about r_0 and identify the restoring force.)

Problem 15.10 ⋆⋆⋆⋆⋆

Two point masses m_1 and m_2, separated by distance d, are released from rest under their mutual gravitational attraction. Find the time for them to collide. (*Hint:* use energy conservation to find $\dot{r}(r)$, then integrate $\mathrm{d}t = \mathrm{d}r/\dot{r}$ with the substitution $r = d\sin^2\eta$.)

Chapter 16

Kepler's Laws and Orbital Mechanics

In the previous chapter we developed Newton's law of universal gravitation and the gravitational potential energy. We now apply these tools to the motion of planets, moons, and satellites: the domain of **orbital mechanics**. The starting point is Kepler's three empirical laws, which Newton showed follow entirely from his law of gravity and the laws of motion.

16.1 Kepler's Three Laws

Johannes Kepler, working for over a decade with Tycho Brahe's unprecedentedly precise observations of Mars, formulated three laws of planetary motion (1609–1619). These were *empirical* regularities—patterns extracted from data, without a theoretical explanation. Half a century later, Newton proved that all three are mathematical consequences of universal gravitation and his laws of motion.

Theorem 16.1: Kepler's Laws

First Law (Law of Ellipses): The orbit of each planet is an ellipse with the Sun at one focus.
Second Law (Law of Equal Areas): The line joining a planet and the Sun sweeps out equal areas in equal time intervals: $\mathrm{d}A/\mathrm{d}t = L/(2m) = \text{const}$.
Third Law (Harmonic Law): The square of the orbital period is proportional to the cube of the semi-major axis:

$$\boxed{T^2 = \frac{4\pi^2}{GM}\,a^3,} \tag{16.1}$$

where M is the central mass and a is the semi-major axis.

We will prove each law in turn, after reviewing the geometry of ellipses.

Geometry of Ellipses

An ellipse is the set of points for which the sum of distances to two fixed points (the **foci**) is constant: $r_1 + r_2 = 2a$. It is described by its **semi-major axis** a (half the longest diameter), **semi-minor axis** $b = a\sqrt{1-e^2}$, and **eccentricity** e $(0 \le e < 1)$. The two foci are at distance $c = ae$ from the center.

The closest and farthest distances from the occupied focus are:

$$r_p = a(1-e) \qquad \text{(periapsis)}, \qquad r_a = a(1+e) \qquad \text{(apoapsis)}. \tag{16.2}$$

Note that $r_p + r_a = 2a$ and $r_p r_a = a^2(1-e^2) = b^2$, which we will use repeatedly.

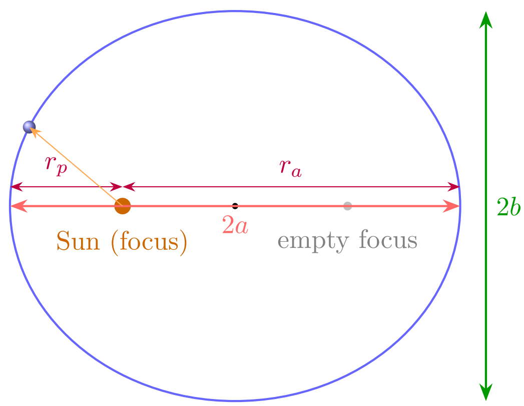


Figure 16.1.1: Geometry of an elliptical orbit. The Sun sits at one focus. The semi-major axis a determines the energy; the eccentricity e determines the shape.

A circle is the special case $e = 0$ ($a = b$, both foci at the center). Earth's orbit is nearly circular ($e = 0.017$), while comets often have $e > 0.99$.

The **semi-latus rectum** $\ell = a(1 - e^2) = b^2/a$ is the distance from the focus to the ellipse measured perpendicular to the major axis. In polar coordinates centered at the occupied focus, the equation of an ellipse is:

$$r(\theta) = \frac{\ell}{1 + e\cos\theta}. \tag{16.3}$$

16.2 Proof of Kepler's Second Law

Of the three laws, the second is the easiest to prove and the most general: it holds for *any* central force, not just gravity.

Proof. Gravity is a central force ($\boldsymbol{F} = F(r)\,\hat{\boldsymbol{r}}$), so the torque about the Sun vanishes: $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{0}$ (since $\boldsymbol{r} \parallel \boldsymbol{F}$). By $\boldsymbol{\tau} = \mathrm{d}\boldsymbol{L}/\mathrm{d}t$, the angular momentum $\boldsymbol{L} = m\boldsymbol{r} \times \boldsymbol{v}$ is conserved. In plane polar coordinates, $L = mr^2\dot{\theta}$.

In a small time interval $\mathrm{d}t$, the radius vector sweeps out a thin triangle with base $r\,\mathrm{d}\theta$ and height r, so the area element is:

$$\mathrm{d}A = \tfrac{1}{2}r^2\,\mathrm{d}\theta. \tag{16.4}$$

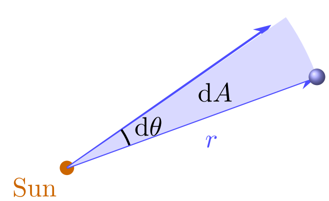


Figure 16.2.1: The area element swept out in time $\mathrm{d}t$: $\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta$.

The areal velocity is therefore:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \text{constant}. \tag{16.5}$$

Since L is conserved, the planet sweeps out area at a constant rate. This is Kepler's second law.

A direct consequence: since $\boldsymbol{v} \perp \boldsymbol{r}$ at both periapsis and apoapsis, $L = mr_p v_p = mr_a v_a$, giving:

$$r_p v_p = r_a v_a. \tag{16.6}$$

The planet moves fastest at periapsis and slowest at apoapsis.

16.3 Proof of Kepler's First Law (The Orbit Equation)

Kepler's first law states that bound orbits are ellipses. To prove this, we must solve the equation of motion for a $1/r^2$ force and show that the solution is a conic section.

Setup. A mass m orbits a mass $M \gg m$ (fixed at the origin). In polar coordinates, the equations of motion are:

$$\text{Radial:} \quad m(\ddot{r} - r\dot{\theta}^2) = -\frac{GMm}{r^2}, \tag{16.7}$$

$$\text{Angular:} \quad \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0 \quad \Longrightarrow \quad L = mr^2\dot{\theta} = \text{const.} \tag{16.8}$$

The angular equation gives us Kepler's second law (already proved). We now focus on the radial equation.

The Binet substitution. Let $u = 1/r$. Using $\dot{\theta} = L/(mr^2) = Lu^2/m$ and the chain rule $\dot{r} = -(L/m)\,\mathrm{d}u/\mathrm{d}\theta$, the radial equation transforms (after some algebra) into:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{GMm^2}{L^2} \equiv \frac{1}{\ell}, \tag{16.9}$$

where we have defined the semi-latus rectum $\ell \equiv L^2/(GMm^2)$.

This is a remarkable result: Eq. (16.9) is the equation of a *simple harmonic oscillator* (which, again, we will get to in more detail in Chapter 18) in θ (with a constant offset $1/\ell$). The general solution is:

$$u(\theta) = \frac{1}{\ell}(1 + e\cos(\theta - \theta_0)),$$

where e and θ_0 are integration constants. Choosing $\theta_0 = 0$ (periapsis at $\theta = 0$) and converting back to $r = 1/u$:

$$\boxed{r(\theta) = \frac{\ell}{1 + e\cos\theta}.} \tag{16.10}$$

This is the polar equation of a **conic section** with focus at the origin:

Orbit type	Eccentricity	Energy
Circle	$e = 0$	$E = -GMm/(2\ell)$
Ellipse	$0 < e < 1$	$E < 0$ (bound)
Parabola	$e = 1$	$E = 0$ (marginally unbound)
Hyperbola	$e > 1$	$E > 0$ (unbound)

For bound orbits ($E < 0$), the orbit is an ellipse with r oscillating between $r_p = \ell/(1 + e)$ and $r_a = \ell/(1 - e)$. The semi-major axis is $a = (r_p + r_a)/2 = \ell/(1 - e^2)$. This completes the proof of Kepler's first law.

Key Point 16.1: The Eccentricity–Energy–Angular Momentum Connection

The eccentricity is determined by both the energy and the angular momentum:

$$e = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}. \tag{16.11}$$

For a given energy $E < 0$, the circular orbit ($e = 0$) has the maximum possible angular momentum. Orbits with less L (at the same E) are more elongated. The degenerate case $L = 0$ gives $e = 1$: radial free fall (a "line orbit").

16.4 Proof of Kepler's Third Law

For circular orbits, the proof is straightforward (equating gravitational and centripetal force gives $T^2 = 4\pi^2 r^3/(GM)$). The extension to ellipses requires more work.

Proof. The total area of an ellipse is $A = \pi ab$. By Kepler's second law, the areal velocity is constant: $\mathrm{d}A/\mathrm{d}t = L/(2m)$. Therefore the period is:

$$T = \frac{A}{\mathrm{d}A/\mathrm{d}t} = \frac{\pi ab}{L/(2m)} = \frac{2\pi mab}{L}. \tag{16.12}$$

Now we express b and L in terms of a and e. We have $b = a\sqrt{1-e^2}$ and $\ell = a(1-e^2) = L^2/(GMm^2)$, so $L = m\sqrt{GMa(1-e^2)}$. Substituting:

$$T = \frac{2\pi m \cdot a \cdot a\sqrt{1-e^2}}{m\sqrt{GMa(1-e^2)}} = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{GMa(1-e^2)}} \tag{16.13}$$

$$= \frac{2\pi a^2}{\sqrt{GMa}} = 2\pi\sqrt{\frac{a^3}{GM}}. \tag{16.14}$$

Therefore:

$$\boxed{T^2 = \frac{4\pi^2}{GM}\, a^3.} \tag{16.15}$$

This is Kepler's third law for elliptical orbits. The period depends *only* on a, not on e—a highly eccentric ellipse and a circle with the same semi-major axis have the same period.

Example 16.1 ("Weighing" the Sun). Earth orbits with $T = 1$ year and $a = 1.496 \times 10^{11}$ m. From Kepler III:

$$M_\odot = \frac{4\pi^2 a^3}{GT^2} = \frac{4\pi^2(1.496 \times 10^{11})^3}{(6.674 \times 10^{-11})(3.156 \times 10^7)^2} = 1.99 \times 10^{30}\,\text{kg}.$$

Kepler's third law lets us "weigh" any body if we can observe something orbiting it.

16.5 Circular Orbits

Circular orbits ($e = 0$) are the simplest case and serve as the foundation for understanding more general orbits.

Orbital Speed and Period

For a satellite of mass m in a circular orbit of radius r around mass M, the gravitational force provides the centripetal acceleration:

$$\frac{GMm}{r^2} = \frac{mv^2}{r}.$$

The mass m cancels (the orbit is independent of the satellite's mass!), giving:

$$\boxed{v_{\text{orb}} = \sqrt{\frac{GM}{r}}, \qquad T = \frac{2\pi r}{v_{\text{orb}}} = 2\pi\sqrt{\frac{r^3}{GM}}.} \tag{16.16}$$

Higher orbits are *slower*: $v \propto 1/\sqrt{r}$.

Energy of a Circular Orbit

The kinetic energy is found by substituting $v^2 = GM/r$:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m \cdot \frac{GM}{r} = \frac{GMm}{2r}.$$

The potential energy is $U = -GMm/r$. The total energy is therefore:

$$\boxed{E = K + U = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}.} \tag{16.17}$$

The total energy is *negative* (the orbit is bound). Three useful relations follow:

$$K = -E = \frac{GMm}{2r}, \qquad U = 2E = -\frac{GMm}{r}, \qquad K = -\tfrac{1}{2}U. \tag{16.18}$$

The last relation—$K = -\frac{1}{2}U$ (time-averaged)—is the **virial theorem** for a $1/r^2$ force. It holds for elliptical orbits as well (with r replaced by a).

Common Mistake 16.1: Higher Orbits Are Slower but Have More Energy

A satellite in a higher orbit moves *slower* ($v \propto 1/\sqrt{r}$) but has *more* total energy ($E = -GMm/(2r)$ becomes less negative as r increases). To move to a higher orbit, you must *add* energy by firing rockets forward, even though the satellite ends up going slower. The paradox resolves because the increase in potential energy exceeds the decrease in kinetic energy: $\Delta U = -2\Delta K$ (from the virial theorem), so $\Delta E = \Delta K + \Delta U = \Delta K - 2\Delta K = -\Delta K > 0$.

Example 16.2 (Geostationary orbit). A geostationary satellite has $T = 24\,\text{hr} = 86\,400\,\text{s}$.
Solution. $r = (GM_E T^2/(4\pi^2))^{1/3} = (3.986 \times 10^{14} \times 86400^2/(4\pi^2))^{1/3} = 4.22 \times 10^7\,\text{m} \approx 6.6\,R_E$. Altitude: $h \approx 35\,800\,\text{km}$. Speed: $v = 2\pi r/T = 3.07\,\text{km/s}$.

Example 16.3 (Low Earth orbit). A satellite at altitude $h = 400\,\text{km}$ (ISS orbit).
Solution. $r = R_E + h = 6.77 \times 10^6\,\text{m}$. $v = \sqrt{GM_E/r} = 7.67\,\text{km/s}$. $T = 2\pi r/v \approx 92\,\text{min}$.

16.6 Elliptical Orbits and Energy

Total Energy of an Elliptical Orbit

We now prove the central result: the total energy of an elliptical orbit depends only on the semi-major axis a.

Proof. At periapsis, the velocity is purely tangential ($\boldsymbol{v} \perp \boldsymbol{r}$), so $L = mr_pv_p$ and:

$$E = \frac{1}{2}mv_p^2 - \frac{GMm}{r_p}. \tag{16.19}$$

Similarly at apoapsis: $L = mr_av_a$ and $E = \frac{1}{2}mv_a^2 - GMm/r_a$. From $r_pv_p = r_av_a$ (angular momentum conservation), we can eliminate v_p and v_a. Using $r_p = a(1-e)$ and $r_a = a(1+e)$:

$$v_p = \frac{L}{mr_p} = \frac{L}{ma(1-e)}, \qquad v_a = \frac{L}{ma(1+e)}.$$

Energy conservation (E is the same at periapsis and apoapsis):

$$\frac{L^2}{2ma^2(1-e)^2} - \frac{GMm}{a(1-e)} = \frac{L^2}{2ma^2(1+e)^2} - \frac{GMm}{a(1+e)}.$$

Rearranging and using $\ell = L^2/(GMm^2) = a(1-e^2)$, this simplifies (after algebra) to:

$$\boxed{E = -\frac{GMm}{2a}.} \tag{16.20}$$

This is a remarkable result: *orbits with the same semi-major axis a but different eccentricities e have the same total energy.* A narrow, highly eccentric ellipse and a circle with the same a are energetically equivalent.

The Vis-Viva Equation

Combining $E = \frac{1}{2}mv^2 - GMm/r$ with $E = -GMm/(2a)$ and solving for v^2:

$$\frac{1}{2}mv^2 = \frac{GMm}{r} - \frac{GMm}{2a} = GMm\left(\frac{1}{r} - \frac{1}{2a}\right).$$

This gives the **vis-viva equation** ("living force," from the 18th-century term for kinetic energy):

$$\boxed{v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right).} \tag{16.21}$$

This relates the orbital speed at any point to the instantaneous distance r and the semi-major axis a. It is the single most useful equation in orbital mechanics.

Special cases:

- **Circular orbit** ($r = a$): $v^2 = GM/a$, recovering Eq. (16.16).
- **Escape** ($a \to \infty$): $v^2 = 2GM/r$, recovering the escape velocity.
- **Periapsis** ($r = r_p = a(1-e)$): $v_p^2 = \frac{GM}{a} \cdot \frac{1+e}{1-e}$.
- **Apoapsis** ($r = r_a = a(1+e)$): $v_a^2 = \frac{GM}{a} \cdot \frac{1-e}{1+e}$.

Example 16.4 (Elliptical orbit speeds). A satellite has perigee altitude 300 km and apogee altitude 3000 km above Earth.

Solution. $r_p = R_E + 300 = 6.67 \times 10^6$ m, $r_a = R_E + 3000 = 9.37 \times 10^6$ m, $a = (r_p + r_a)/2 = 8.02 \times 10^6$ m.

Vis-viva at perigee: $v_p = \sqrt{GM_E(2/r_p - 1/a)} = 8.35$ km/s. At apogee: $v_a = \sqrt{GM_E(2/r_a - 1/a)} = 5.95$ km/s.

Check angular momentum: $r_p v_p = 5.57 \times 10^{10} = r_a v_a$. ✓

16.7 Hohmann Transfer Orbits

The most fuel-efficient two-impulse maneuver to transfer between two coplanar circular orbits (radii r_1 and $r_2 > r_1$) is the **Hohmann transfer**: an elliptical orbit tangent to both circular orbits. The transfer ellipse has periapsis at r_1 and apoapsis at r_2, so:

$$a_t = \frac{r_1 + r_2}{2}. \tag{16.22}$$

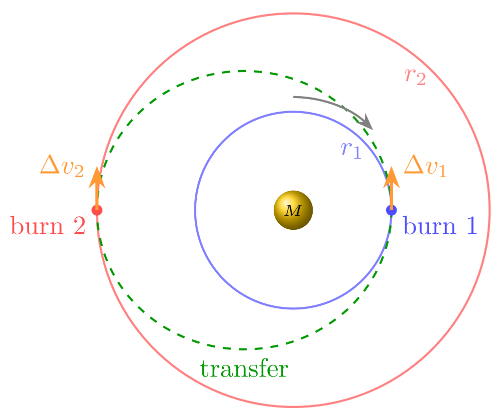


Figure 16.7.1: Hohmann transfer between circular orbits of radii r_1 and r_2.

Derivation of the Δv budget. Before the first burn, the satellite is in a circular orbit at r_1 with speed $v_1 = \sqrt{GM/r_1}$. After the burn, it must be on the transfer ellipse at $r = r_1$ (periapsis), with speed given by vis-viva:

$$v_{t1} = \sqrt{GM\left(\frac{2}{r_1} - \frac{1}{a_t}\right)} = \sqrt{\frac{GM}{r_1} \cdot \frac{2r_2}{r_1 + r_2}}.$$

The first velocity kick is therefore:

$$\Delta v_1 = v_{t1} - v_1 = \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right). \tag{16.23}$$

At the top of the transfer ellipse ($r = r_2$, apoapsis), the satellite arrives with speed $v_{t2} = \sqrt{GM(2/r_2 - 1/a_t)}$ and must be boosted to the circular speed $v_2 = \sqrt{GM/r_2}$:

$$\Delta v_2 = v_2 - v_{t2} = \sqrt{\frac{GM}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right). \tag{16.24}$$

The transfer time is half the period of the transfer ellipse: $t = \pi\sqrt{a_t^3/(GM)}$.

Example 16.5 (LEO to geostationary). $r_1 = R_E + 200$ km $= 6.57\times10^6$ m, $r_2 = 4.22\times10^7$ m, $a_t = 2.44\times10^7$ m.

Solution. $v_1 = \sqrt{GM_E/r_1} = 7.78\,\text{km/s}$. $v_{t1} = \sqrt{GM_E(2/r_1 - 1/a_t)} = 10.24\,\text{km/s}$. $\Delta v_1 = 2.46\,\text{km/s}$.

$v_2 = \sqrt{GM_E/r_2} = 3.07\,\text{km/s}$. $v_{t2} = \sqrt{GM_E(2/r_2 - 1/a_t)} = 1.60\,\text{km/s}$. $\Delta v_2 = 1.47\,\text{km/s}$.

Total $\Delta v = 3.93\,\text{km/s}$. Transfer time $= \pi\sqrt{(2.44\times10^7)^3/(3.986\times10^{14})} \approx 5.3\,\text{hr}$.

16.8 Tidal Forces

The tidal force arises from the variation of the gravitational field across an extended object. Unlike the gravitational force itself (which scales as $1/r^2$), tidal effects scale as $1/r^3$ and become dramatic near compact objects.

Derivation of the Tidal Acceleration

Consider two small masses separated by Δr (radially) at distance r from a point mass M. The gravitational accelerations at the two locations are $g(r) = GM/r^2$ and $g(r+\Delta r) = GM/(r+\Delta r)^2$. The *tidal acceleration*—the difference—is:

$$\begin{aligned}\Delta a = g(r) - g(r+\Delta r) &= \frac{GM}{r^2} - \frac{GM}{(r+\Delta r)^2} \\ &= \frac{GM}{r^2}\left[1 - \frac{1}{(1+\Delta r/r)^2}\right] \approx \frac{GM}{r^2}\cdot\frac{2\Delta r}{r} = \frac{2GM}{r^3}\,\Delta r,\end{aligned} \tag{16.25}$$

where the last step uses the binomial approximation $(1+x)^{-2} \approx 1 - 2x$ for $\Delta r \ll r$. Therefore:

$$\boxed{\Delta a = \frac{2GM}{r^3}\,\Delta r.} \tag{16.26}$$

The key feature is the $1/r^3$ scaling: tidal forces grow much more rapidly than the gravitational force itself as you approach the source.

Tidal forces are responsible for: ocean tides on Earth (due to the Moon and Sun), the synchronous rotation of the Moon (one face always toward Earth), the volcanic activity of Jupiter's moon Io (heated by tidal flexing), and the spectacular "spaghettification" of objects falling into neutron stars or black holes.

The Roche Limit

A fluid satellite held together only by self-gravity will be torn apart if the tidal force across it exceeds its self-gravitational cohesion. The critical distance is the **Roche limit**:

$$d_{\text{Roche}} \approx 2.44\,R_M\left(\frac{\rho_M}{\rho_s}\right)^{1/3}, \tag{16.27}$$

where R_M and ρ_M are the primary's radius and density, and ρ_s is the satellite's density.

Rough derivation. Consider a small mass δm on the surface of a fluid satellite of mass m_s and radius R_s at distance d from a primary of mass M. The tidal force pulling δm away from the satellite is $\sim 2GMR_s\delta m/d^3$. The self-gravitational force holding δm to the satellite is $\sim Gm_s\delta m/R_s^2$. Tidal disruption occurs when these are comparable:

$$\frac{2GMR_s}{d^3} \sim \frac{Gm_s}{R_s^2} \quad \Longrightarrow \quad d^3 \sim \frac{2MR_s^3}{m_s} = \frac{2\rho_M R_M^3}{\rho_s}.$$

The exact calculation (accounting for the satellite's deformation) gives $d_{\text{Roche}} \approx 2.44\, R_M(\rho_M/\rho_s)^{1/3}$.

Saturn's rings lie within Saturn's Roche limit—ring material cannot coalesce into a moon because tidal forces overwhelm self-gravity.

Example 16.6 (Earth's Roche limit). For a rocky satellite ($\rho_s = 2000\,\text{kg/m}^3$): $d_{\text{Roche}} = 2.44(6370)(5500/2000)^{1/3} = 21\,800\,\text{km} \approx 3.4\,R_E$. The Moon (at $60\,R_E$) is far outside.

More Worked Examples

Example 16.7 (Earth's orbital speed variation). $a = 1.496 \times 10^{11}\,\text{m}$, $e = 0.0167$. $r_p = a(1-e) = 1.471 \times 10^{11}\,\text{m}$, $r_a = a(1+e) = 1.521 \times 10^{11}\,\text{m}$.

Solution. Vis-viva: $v_p = \sqrt{GM_\odot(2/r_p - 1/a)} = 30.3\,\text{km/s}$, $v_a = \sqrt{GM_\odot(2/r_a - 1/a)} = 29.3\,\text{km/s}$ (3% variation—Earth's orbit is very nearly circular).

Problems

Problem 16.1 ⋆⋆⋆

A geostationary satellite orbits Earth with $T = 24\,\text{hr}$. (a) Find its orbital radius using Kepler's third law. (b) Find its orbital speed. (c) What is the altitude above Earth's surface? Use $M_E = 5.97 \times 10^{24}\,\text{kg}$, $R_E = 6.37 \times 10^6\,\text{m}$.

Problem 16.2 ⋆⋆⋆

Using the vis-viva equation, find the speed of Earth at perihelion and aphelion. Earth's orbit has $a = 1.496 \times 10^{11}\,\text{m}$ and $e = 0.0167$. Use $M_\odot = 1.989 \times 10^{30}\,\text{kg}$.

Problem 16.3 ⋆⋆⋆

Two satellites orbit a planet of radius $R_p = 9 \times 10^6\,\text{m}$. Satellite A: $m_A = 68\,\text{kg}$, $r_A = 7 \times 10^7\,\text{m}$, $v_A = 4800\,\text{m/s}$. Satellite B: $m_B = 84\,\text{kg}$, $r_B = 3 \times 10^7\,\text{m}$. (a) Find v_B. (b) Find both orbital periods. (c) Verify that $T_A^2/T_B^2 = (r_A/r_B)^3$.

Problem 16.4 ⋆⋆⋆⋆

A satellite in a circular orbit of radius $r_1 = R_E + 300\,\text{km}$ transfers to geostationary orbit ($r_2 = R_E + 35\,800\,\text{km}$) via a Hohmann transfer. (a) Find the semi-major axis of the transfer ellipse. (b) Find Δv_1 and Δv_2. (c) Find the total Δv and the transfer time.

Problem 16.5 ⋆⋆⋆⋆⋆

A binary star system: two stars, each of mass M, in circular orbits about their common CM with separation $2d$. (a) Find the orbital speed and period. (b) Find the total energy. (c) Show that a small radial displacement from the midpoint is unstable. (d) Show that a small perpendicular displacement leads to simple harmonic oscillation and find the angular frequency ω. (*Hint:* "simple harmonic" means the restoring force is proportional to the displacement, $F \approx -k_{\text{eff}}\,\epsilon$. The angular frequency of the resulting oscillation is $\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}$; see Chapter 18 for a full treatment.)

Problem 16.6 ⋆⋆⋆⋆

The Moon has $r_M = 3.84 \times 10^8\,\text{m}$ and $T_M = 27.3\,\text{days}$. (a) Compute GM_E from this data. (b) Find the period of a satellite at altitude 200 km. (c) At what altitude is $T = 90\,\text{min}$?

Problem 16.7 ⋆⋆⋆

A satellite orbits Earth with perigee altitude 500 km and apogee altitude 5000 km. (a) Find the semi-major axis and eccentricity. (b) Find speeds at perigee and apogee. (c) Find the orbital period.

Problem 16.8 ⋆⋆⋆⋆

A spacecraft in a circular orbit of radius r_0 fires its engines, increasing its speed tangentially by Δv. (a) Find the semi-major axis of the new elliptical orbit. (b) Find the apoapsis distance. (c) Show that for $\Delta v \ll v_{\text{orb}}$: $\Delta r_a \approx 4r_0(\Delta v/v_{\text{orb}})$.

Problem 16.9 ⋆⋆⋆

An astronaut ($h = 1.8\,\text{m}$) orbits a neutron star ($M = 1.4M_\odot$, $R = 10\,\text{km}$) at distance r. (a) Find the tidal acceleration across the body as a function of r. (b) At what r does this equal $g = 9.8\,\text{m/s}^2$? (c) Compare to the Roche limit for $\rho \approx 1000\,\text{kg/m}^3$.

Problem 16.10 ⋆⋆⋆⋆⋆

(*Weighing the Sun.*) Using $a = 1.496 \times 10^{11}\,\mathrm{m}$, $T = 1$ year, $G = 6.674 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$: (a) Compute $M_\odot$. (b) An asteroid has $T = 5.0$ years. Find its semi-major axis in AU. (c) Jupiter's moon Io has $T = 1.77\,\mathrm{days}$ and $a = 4.22 \times 10^8\,\mathrm{m}$. Find M_{Jupiter}.

Chapter 17

Black Holes

Black holes are among the most extraordinary objects in the universe—regions of spacetime where gravity is so strong that nothing, not even light, can escape. While a complete treatment requires Einstein's general theory of relativity, the essential features of black holes can be motivated (and in several cases correctly predicted) using the Newtonian concepts developed in this book.

The idea that gravity could prevent light from escaping is surprisingly old. In 1783, the English clergyman and natural philosopher John Michell reasoned that a sufficiently massive and compact star would have an escape velocity exceeding the speed of light, rendering it invisible. Independently, Pierre-Simon Laplace published a similar calculation in 1796. These "dark stars" were largely forgotten until general relativity made them inevitable: Karl Schwarzschild found the exact solution for the gravitational field outside a spherical mass in 1916, just weeks after Einstein published his field equations.

In this chapter we explore what Newtonian mechanics can tell us about these extreme objects, and we will be careful to flag where the Newtonian picture breaks down and general relativity takes over.

17.1 The Schwarzschild Radius

The escape velocity from the surface of a body of mass M and radius R is $v_e = \sqrt{2GM/R}$ (Chapter 15). As R decreases at fixed M, v_e increases. A natural question: at what radius does v_e equal the speed of light c?

Setting $v_e = c$:

$$c = \sqrt{\frac{2GM}{R_s}} \implies \boxed{R_s = \frac{2GM}{c^2}.} \tag{17.1}$$

This is the **Schwarzschild radius**. Any object compressed below its Schwarzschild radius becomes a black hole.

Common Mistake 17.1: Newtonian Derivation Caveat

The Newtonian derivation gives the correct Schwarzschild radius, but the reasoning is not rigorous: it treats light as a massive particle obeying $\frac{1}{2}mc^2 = GMm/R$, whereas photons are massless and do not obey Newtonian mechanics. The rigorous derivation requires solving Einstein's field equations. Nevertheless, the Newtonian result provides the correct scale and excellent physical intuition.

A useful scaling relation: since $R_s = 2GM/c^2$, the Schwarzschild radius is proportional to mass:

$$R_s \approx 3.0\,\text{km} \times \left(\frac{M}{M_\odot}\right). \tag{17.2}$$

A solar-mass black hole has $R_s \approx 3\,\text{km}$; a $10M_\odot$ black hole has $R_s \approx 30\,\text{km}$.

Example 17.1 (Schwarzschild radii). (a) Sun: $R_s = 2(6.674\times10^{-11})(2.0\times10^{30})/(3\times10^8)^2 = 2960\,\text{m} \approx 3.0\,\text{km}$. (b) Earth: $R_s \approx 8.9\,\text{mm}$. (c) $10M_\odot$: $R_s = 30\,\text{km}$—about the size of a city.

Average Density of a Black Hole

The "average density" of a black hole (mass M distributed within volume $\frac{4}{3}\pi R_s^3$) is:

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R_s^3} = \frac{3c^6}{32\pi G^3 M^2} \propto \frac{1}{M^2}. \tag{17.3}$$

For a $10M_\odot$ black hole: $\bar{\rho} \sim 10^{17}$ kg/m^3—comparable to nuclear density. But for a supermassive black hole ($M \sim 10^9 M_\odot$): $\bar{\rho} \sim 1$ kg/m^3—less than air! Counterintuitively, the most massive black holes are the *least* dense (on average). You could form one by uniformly distributing matter at the density of water over a sufficiently large volume.

17.2 The Event Horizon

The spherical surface at $r = R_s$ is called the **event horizon**. It marks the boundary of the black hole: the "point of no return" beyond which escape is impossible.

Several features are worth emphasizing:

- The event horizon is *not* a physical surface—there is no wall, membrane, or barrier. An observer freely falling through it would not notice anything locally special at the moment of crossing (for a sufficiently large black hole where tidal forces are weak).
- Once inside, *all* trajectories lead to the singularity at $r = 0$. In general relativity, the coordinate r inside the horizon plays the role of a time coordinate: moving toward $r = 0$ is as inevitable as moving forward in time.
- From the perspective of a distant observer, an object falling toward the horizon appears to slow down, redden, and dim: it never appears to cross. This is a consequence of gravitational time dilation: clocks near the horizon run slower (from the distant observer's viewpoint).

17.3 Black Hole Formation

How does an object end up compressed within its Schwarzschild radius? The answer lies in stellar evolution.

A normal star is supported against gravitational collapse by thermal pressure from nuclear fusion in its core. When the fuel is exhausted, the core contracts. What happens next depends on the core mass:

- $M_{\mathbf{core}} \lesssim 1.4\,M_\odot$ (the **Chandrasekhar limit**): Electron degeneracy pressure—a quantum-mechanical effect arising from the Pauli exclusion principle—halts the collapse. The result is a **white dwarf**, with radius $\sim R_\oplus$ and density $\sim 10^9$ kg/m^3.
- $1.4\,M_\odot \lesssim M_{\mathbf{core}} \lesssim 3\,M_\odot$: Electron degeneracy is overwhelmed. Electrons and protons merge into neutrons, and neutron degeneracy pressure halts the collapse. The result is a **neutron star**: $R \sim 10\,\text{km}$, $\rho \sim 10^{17}$ kg/m^3, spinning up to hundreds of times per second.
- $M_{\mathbf{core}} \gtrsim 3\,M_\odot$: No known force can halt the collapse. The core contracts through its own Schwarzschild radius and forms a **black hole**.

The mass limits above are approximate and remain active areas of research. The key Newtonian insight is that self-gravity grows without bound as an object contracts ($F \propto 1/r^2$), and if no force can resist it, collapse to a black hole is inevitable.

17.4 Orbits Near a Black Hole

In Chapter 16 we studied circular orbits at any radius $r > R$ outside a spherical body. Near a black hole, something new happens: not all circular orbits are stable.

The Innermost Stable Circular Orbit (ISCO)

Recall from Problem 15.9 that the effective potential for radial motion is:

$$V_{\text{eff}}(r) = -\frac{GMm}{r} + \frac{L^2}{2mr^2}. \tag{17.4}$$

The first term (gravity, attractive) and the second (centrifugal barrier, repulsive) combine to create a potential well. Circular orbits sit at the minimum of V_{eff}, and they are stable because a small radial perturbation produces a restoring force (the orbit oscillates about the minimum).

In Newtonian gravity, this minimum exists at *every* radius, all circular orbits are stable. In general relativity, however, the effective potential is modified and acquires an additional attractive term at small r that destroys the potential minimum inside a critical radius. The result is the **innermost stable circular orbit** (ISCO):

$$r_{\text{ISCO}} = 3R_s = \frac{6GM}{c^2} \qquad \text{(non-spinning black hole).} \tag{17.5}$$

Inside this radius, no stable circular orbit exists. Any particle perturbed inward from the ISCO spirals into the black hole.

Using the Newtonian orbital speed at the ISCO ($r = 6GM/c^2$):

$$v_{\text{ISCO}} = \sqrt{\frac{GM}{r}} = \sqrt{\frac{c^2}{6}} = \frac{c}{\sqrt{6}} \approx 0.41c.$$

The material at the ISCO moves at 41% of the speed of light—deep in the relativistic regime, underscoring that Newtonian mechanics is only an approximation here.

The Photon Sphere

At $r = \frac{3}{2}R_s = 3GM/c^2$, photons can orbit the black hole in an unstable circular orbit (the "photon sphere"). A Newtonian estimate gives $v_{\text{orb}} = \sqrt{GM/r} = c/\sqrt{3} \approx 0.58c$ at this radius, but this is a case where Newtonian mechanics is inadequate: the correct general-relativistic treatment gives photons traveling at c (as always) on null geodesics at exactly $r = 3GM/c^2$.

The photon sphere is responsible for the bright ring of light seen in the Event Horizon Telescope images of supermassive black holes (M87* and Sgr A*).

17.5 Tidal Forces and Spaghettification

Near a black hole, the gravitational field changes rapidly with distance. The tidal acceleration between two points separated by Δr at distance r from a mass M is (from Eq. (16.26) of Chapter 16):

$$\Delta a = \frac{2GM}{r^3}\,\Delta r. \tag{17.6}$$

An infalling object is stretched radially (head-to-foot) and compressed laterally (side-to-side)—a process vividly called **spaghettification**.

Example 17.2 (Tidal forces at the horizon). At the event horizon ($r = R_s = 2GM/c^2$), the tidal acceleration across a body of length Δr is:

$$\Delta a = \frac{2GM\,\Delta r}{(2GM/c^2)^3} = \frac{c^6\,\Delta r}{4G^2M^2}.$$

This scales as $1/M^2$: smaller black holes have *stronger* tidal forces at the horizon.

For a person ($\Delta r = 2\,\text{m}$) at the horizon of a $10M_\odot$ black hole ($M = 2 \times 10^{31}$ kg):

$$\Delta a = \frac{(3\times10^8)^6(2)}{4(6.674\times10^{-11})^2(2\times10^{31})^2} = \frac{1.46\times10^{51}}{7.13\times10^{42}} \approx 2.0\times10^8\,\text{m/s}^2 \approx 2\times10^7\,g.$$

This is instantly lethal: the person is spaghettified long before reaching the horizon.

For a supermassive black hole ($M = 4\times10^6 M_\odot$): $\Delta a \propto 1/M^2$ gives $\Delta a \approx 1.3\times10^{-3}\,\text{m/s}^2 \approx 10^{-4}\,g$. This is barely noticeable! A person could survive crossing the event horizon of a supermassive black hole with no ill effects, a remarkable and counterintuitive result.

17.6 Gravitational Redshift

A photon emitted near a massive object loses energy as it climbs out of the gravitational potential well. We can estimate the effect using energy conservation, treating the photon's energy $E = h\nu$ as analogous to kinetic energy.

Derivation. Consider a photon emitted at radius r from a mass M with frequency ν_{emit}. At infinity, its frequency is ν_∞. Energy conservation gives:

$$h\nu_\infty = h\nu_{\text{emit}} - \frac{GM(h\nu_{\text{emit}}/c^2)}{r},$$

where we have used $m_{\text{eff}} = h\nu/c^2$ (from $E = mc^2$) as the photon's effective gravitational mass. Rearranging:

$$\frac{\Delta\nu}{\nu} = \frac{\nu_{\text{emit}} - \nu_\infty}{\nu_{\text{emit}}} = \frac{GM}{rc^2} = \frac{R_s}{2r}. \tag{17.7}$$

The fractional frequency decrease equals the gravitational potential (in units of c^2). This is the **gravitational redshift**.

Key Point 17.1: Gravitational Redshift and Time Dilation

Since a photon's frequency is the tick rate of an electromagnetic oscillator, gravitational redshift is equivalent to **gravitational time dilation**: clocks in stronger gravitational fields run slower. This effect has been measured to high precision using atomic clocks at different altitudes, and it must be corrected for in the GPS satellite system (whose clocks are in weaker gravity than ground-based clocks).

As $r \to R_s$: $\Delta\nu/\nu \to 1/2$ in the Newtonian approximation (the exact GR result is $\Delta\nu/\nu \to \infty$—the redshift diverges at the horizon, and photons are infinitely redshifted). An object falling toward a black hole appears to a distant observer to redden, dim, and freeze at the horizon.

Example 17.3 (Redshift from a neutron star vs. the Sun). A neutron star ($M = 1.4M_\odot$, $R = 10\,\text{km}$):

$$\frac{\Delta\nu}{\nu} = \frac{GM}{Rc^2} = \frac{(6.674\times10^{-11})(2.8\times10^{30})}{(10^4)(9\times10^{16})} = 0.21.$$

The light is redshifted by 21%—a substantial and easily measurable effect.

For the Sun ($R_\odot = 6.96 \times 10^8$ m): $\Delta\nu/\nu = GM_\odot/(R_\odot c^2) = 2.1 \times 10^{-6}$—only 2 parts per million, but measured by Pound and Rebka in 1959 (in an equivalent terrestrial experiment) and confirmed to 1% accuracy.

17.7 Observational Evidence for Black Holes

Black holes cannot be seen directly (by definition), but their gravitational influence on surrounding matter is unmistakable. The principal lines of evidence include:

- **X-ray binaries:** A black hole in a binary system strips gas from its companion star. The gas spirals inward through an *accretion disk*, heating to millions of kelvins and emitting intense X-rays. The system Cygnus X-1, identified in 1972, was the first widely accepted stellar-mass black hole candidate.
- **Stellar orbits around Sgr A*:** Stars near the center of the Milky Way have been tracked for decades, revealing orbits around an unseen mass of $\sim 4 \times 10^6 M_\odot$ confined within a region smaller than the Solar System. Andrea Ghez and Reinhard Genzel shared the 2020 Nobel Prize in Physics for this work.
- **Gravitational waves:** The merger of two black holes produces gravitational waves—ripples in spacetime predicted by general relativity. LIGO made the first direct detection in September 2015 (published 2016), observing the merger of two black holes of approximately $36M_\odot$ and $29M_\odot$ into a $62M_\odot$ remnant. The "missing" $3M_\odot$ was radiated as gravitational-wave energy ($E = \Delta m\, c^2$).
- **The Event Horizon Telescope (EHT):** In 2019, the EHT collaboration released the first image of a black hole—the supermassive black hole M87* ($M \approx 6.5 \times 10^9 M_\odot$). The image shows a bright ring of emission (from the photon sphere and accretion disk) surrounding a dark "shadow"—the silhouette of the event horizon.

These observations confirm that objects with the predicted properties of black holes exist throughout the universe, from stellar-mass remnants to supermassive monsters at the centers of galaxies.

Example 17.4 (Schwarzschild radius of M87*). $M = 6.5 \times 10^9 M_\odot = 1.3 \times 10^{40}$ kg. $R_s = 2GM/c^2 = 2(6.674 \times 10^{-11})(1.3 \times 10^{40})/(9 \times 10^{16}) = 1.9 \times 10^{13}\,\text{m} \approx 130\,\text{AU}$. This is larger than the orbit of Pluto. The shadow seen by the EHT is roughly $2.6\, R_s$ in diameter (a general-relativistic prediction that has been confirmed observationally).

Problems

Problem 17.1 ⋆⋆

Compute the Schwarzschild radius $R_s = 2GM/c^2$ for (a) the Sun ($M_\odot = 2.0 \times 10^{30}$ kg), (b) the Earth ($M_E = 6.0 \times 10^{24}$ kg), and (c) a $10\,\mathrm{M}_\odot$ stellar-mass black hole. Express each answer in convenient units and compare to a familiar object of similar size.

Problem 17.2 ⋆⋆⋆

The supermassive black hole Sgr A* at the center of the Milky Way has mass $M \approx 4 \times 10^6\, M_\odot$. (a) Find its Schwarzschild radius and express it in AU ($1\,\mathrm{AU} \approx 1.5 \times 10^{11}$ m). Compare to the size of Mercury's orbit (~ 0.39 AU). (b) Find the average density assuming the mass is uniformly distributed within R_s. (c) Compare this density to that of water ($1000\,\mathrm{kg/m^3}$). Is the result what you expected?

Problem 17.3 ⋆⋆⋆⋆

An astronaut of height $\Delta r = 2.0$ m falls feet-first toward a black hole of mass M. The tidal acceleration across the astronaut's body at distance r from the center is $\Delta a = 2GM\Delta r/r^3$. (a) Show that at the event horizon ($r = R_s$), the tidal acceleration simplifies to $\Delta a = c^6 \Delta r/(4G^2M^2)$ and thus scales as $1/M^2$. (b) Compute Δa at the horizon for a $10\,\mathrm{M}_\odot$ stellar-mass black hole. Express in units of g. (c) Compute Δa at the horizon for the $4 \times 10^6\, M_\odot$ Sgr A*. Could the astronaut survive crossing this horizon? (d) Explain physically why larger black holes are "gentler" at the horizon.

Problem 17.4 ⋆⋆

The "average density" of a black hole is defined as $\bar{\rho} = M/(\frac{4}{3}\pi R_s^3)$. (a) Express $\bar{\rho}$ in terms of fundamental constants and M, and show that $\bar{\rho} \propto 1/M^2$. (b) Compute $\bar{\rho}$ for a $10M_\odot$ black hole and compare to nuclear density ($\sim 2 \times 10^{17}\,\mathrm{kg/m^3}$). (c) What mass M (in solar masses) would give $\bar{\rho}$ equal to the density of water?

Problem 17.5 ⋆⋆⋆

A collapsing star of mass M contracts from its initial radius R toward its Schwarzschild radius. At some intermediate radius r, the escape velocity reaches half the speed of light. (a) Find this radius in terms of R_s. (b) Express this radius in km for a $10M_\odot$ star. (c) What fraction of c is the escape velocity at $r = 2R_s$?

Problem 17.6 ⋆⋆⋆

Light emitted from the surface of a compact object at radius R from a mass M is gravitationally redshifted by a fractional amount $\Delta\nu/\nu \approx GM/(Rc^2)$. (a) Compute the gravitational redshift for light emitted from the surface of a neutron star ($M = 1.4\, M_\odot$, $R = 10$ km). Show all intermediate steps. (b) Compute the gravitational redshift for light emitted from the Sun's surface ($M_\odot = 2.0 \times 10^{30}$ kg, $R_\odot = 6.96 \times 10^8$ m). (c) By what factor is the neutron star's redshift larger? Why is this effect so much more pronounced?

Problem 17.7 ⋆⋆⋆⋆

For a non-spinning (Schwarzschild) black hole, the innermost stable circular orbit (ISCO) is at $r_{\mathrm{ISCO}} = 3R_s = 6GM/c^2$. Inside this radius, no stable circular orbit exists and matter spirals inward. (a) Using the Newtonian formula $v = \sqrt{GM/r}$, find the orbital speed at the ISCO and express it as a fraction of c. (b) Find the orbital period at the ISCO for a $10M_\odot$ black hole ($M = 2.0 \times 10^{31}$ kg).

Express in milliseconds. (c) The accretion disk around a black hole terminates at the ISCO. Find the ISCO radius in km for the $4 \times 10^6\, M_\odot$ Sgr A*.

Problem 17.8 ⋆⋆⋆

From $R_s = 2GM/c^2$, find the mass of a black hole whose Schwarzschild radius equals: (a) 1 AU (1.5×10^{11} m). Express in solar masses. (b) The radius of the observable universe ($\sim 4.4 \times 10^{26}$ m). Express in kg and in solar masses. (c) The observable universe contains roughly 10^{53} kg of ordinary matter. Compare this to your answer in (b) and comment.

Problem 17.9 ⋆⋆⋆⋆

(*Beyond Newtonian mechanics.*) Hawking showed that a black hole radiates thermally with temperature $T_H = \hbar c^3/(8\pi GMk_B)$, where $\hbar = 1.055 \times 10^{-34}$ J s and $k_B = 1.381 \times 10^{-23}$ J/K. (a) Compute T_H for a solar-mass black hole. Compare to the temperature of the cosmic microwave background (2.7 K). (b) Show that $T_H \propto 1/M$: smaller black holes are hotter. (c) The luminosity of a blackbody is $L = \sigma_{\text{SB}} A T^4$, where $A = 4\pi R_s^2$ and $\sigma_{\text{SB}} = 5.67 \times 10^{-8}\,\text{W}\,\text{m}^{-2}\,\text{K}^{-4}$. Show that $L \propto 1/M^2$ and that the evaporation timescale (time for the black hole to radiate away all its energy Mc^2) scales as $t_{\text{evap}} \propto M^3$.

Problem 17.10 ⋆⋆⋆

The "surface gravity" of a black hole is defined as $g_s = GM/R_s^2$. (a) Show that $g_s = c^4/(4GM)$. (b) Compute g_s for a solar-mass black hole and express in units of Earth's surface gravity $g_\oplus = 9.8\,\text{m/s}^2$. (c) Show that $g_s \propto 1/M$: more massive black holes have weaker surface gravity. Is this consistent with what you found for tidal forces in Problem 17.3?

Chapter 18

Harmonic Motion

Oscillatory motion (motion that repeats periodically about a stable equilibrium) is one of the most ubiquitous phenomena in physics. A mass on a spring, a swinging pendulum, the vibrations of a guitar string, the oscillation of atoms in a crystal lattice, the alternating current in an electrical circuit, all are governed by the same mathematical framework.

This universality has a simple mathematical origin: near any stable equilibrium, the restoring force is approximately proportional to the displacement, regardless of the detailed physics. This leads to **simple harmonic motion** (SHM), the subject of this chapter. We will also study what happens when friction (damping) and external driving forces are present, leading to the phenomena of exponential decay and resonance.

18.1 Simple Harmonic Motion

Definition 18.1: Simple Harmonic Motion

A system undergoes **simple harmonic motion** when its equation of motion takes the form:

$$\boxed{\ddot{x} + \omega_0^2 x = 0,} \tag{18.1}$$

where ω_0 is the **angular frequency** (in rad/s). The general solution is:

$$\boxed{x(t) = A\cos(\omega_0 t + \phi),} \tag{18.2}$$

where A is the **amplitude** (maximum displacement) and ϕ is the **phase constant**, both determined by initial conditions. (For a derivation of this solution, see Appendix A.6.)

The period T, frequency f, and angular frequency ω_0 are related by:

$$T = \frac{2\pi}{\omega_0} = \frac{1}{f}, \qquad f = \frac{\omega_0}{2\pi}, \qquad \omega_0 = 2\pi f = \frac{2\pi}{T}. \tag{18.3}$$

The Mass–Spring System

The prototypical SHM system is a block of mass m attached to a spring of constant k on a frictionless surface.

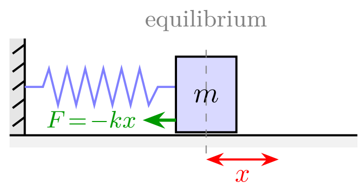


Figure 18.1.1: A mass–spring system: displacement x from equilibrium produces the restoring force $F = -kx$.

Derivation. Newton's second law gives $ma = F = -kx$, or:

$$m\ddot{x} = -kx \quad \Longrightarrow \quad \ddot{x} + \frac{k}{m}x = 0.$$

This is the SHM equation with:

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}}, \qquad T = 2\pi\sqrt{\frac{m}{k}}.} \tag{18.4}$$

The period depends on m and k but *not* on the amplitude A—this is a hallmark of SHM and is called **isochronism**.

Velocity, Acceleration, and Initial Conditions

Differentiating $x(t) = A\cos(\omega_0 t + \phi)$:

$$v(t) = \dot{x} = -A\omega_0 \sin(\omega_0 t + \phi), \tag{18.5}$$
$$a(t) = \ddot{x} = -A\omega_0^2 \cos(\omega_0 t + \phi) = -\omega_0^2 x(t). \tag{18.6}$$

The maximum speed is $v_{\max} = A\omega_0$ (at $x = 0$); the maximum acceleration is $a_{\max} = A\omega_0^2$ (at $x = \pm A$).

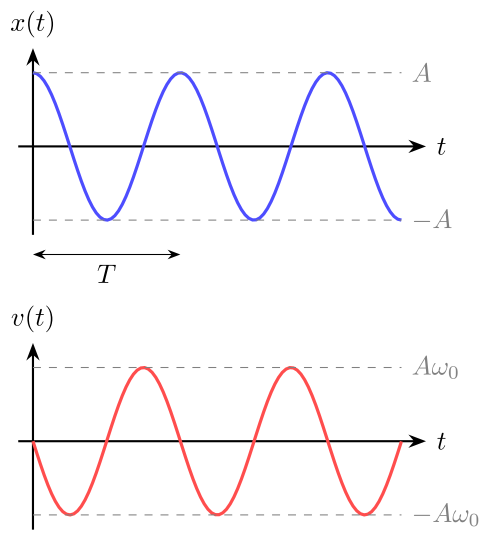


Figure 18.1.2: Position and velocity for SHM with $\phi = 0$. The velocity leads the position by a quarter period (90°).

The amplitude A and phase ϕ are determined by the initial conditions $x(0) = x_0$ and $v(0) = v_0$:

$$A = \sqrt{x_0^2 + (v_0/\omega_0)^2}, \qquad \tan\phi = -\frac{v_0}{\omega_0 x_0}. \tag{18.7}$$

Special cases: released from rest at x_0 gives $A = x_0$, $\phi = 0$; launched from equilibrium at speed v_0 gives $A = v_0/\omega_0$, $\phi = -\pi/2$.

Energy in SHM

The kinetic energy is $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \phi) = \frac{1}{2}kA^2 \sin^2(\omega_0 t + \phi)$.

The potential energy is $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$.

The total energy is:

$$\boxed{E = K + U = \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0^2 A^2 = \text{constant.}} \tag{18.8}$$

Energy is conserved (no friction), continuously converting between kinetic and potential. At $x = 0$: all kinetic. At $x = \pm A$: all potential. The speed at position x follows from energy conservation:

$$v = \omega_0\sqrt{A^2 - x^2}. \tag{18.9}$$

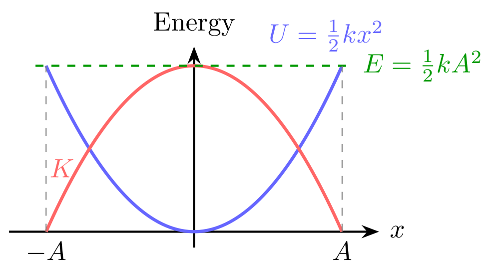


Figure 18.1.3: Energy in SHM as a function of position. The potential energy $U = \frac{1}{2}kx^2$ (blue) and kinetic energy $K = E - U$ (red) sum to a constant total energy $E = \frac{1}{2}kA^2$ (dashed green).

> **Key Point 18.1: Why SHM Is Universal**
>
> Near any stable equilibrium x_0, the potential energy can be Taylor-expanded: $U(x) \approx U(x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2 + \cdots$ (the linear term vanishes because $U'(x_0) = 0$ at equilibrium). The restoring force is $F = -U'(x) \approx -U''(x_0)(x - x_0) = -k_{\text{eff}}\,\xi$, which is Hooke's law with $k_{\text{eff}} = U''(x_0)$. Therefore *any* system near a stable equilibrium undergoes SHM, with $\omega_0 = \sqrt{k_{\text{eff}}/m}$. This is why the SHM equation appears in such a wide variety of physical contexts.

Example 18.1 (Mass on a spring). A block of mass $m = 0.50\,\text{kg}$ on a spring ($k = 32\,\text{N/m}$) is pulled to $x_0 = 0.15\,\text{m}$ and released from rest.

Solution. $\omega_0 = \sqrt{32/0.50} = 8.0\,\text{rad/s}$. $T = 2\pi/8 = 0.785\,\text{s}$. $v_{\max} = A\omega_0 = 0.15(8) = 1.2\,\text{m/s}$. $E = \frac{1}{2}kA^2 = \frac{1}{2}(32)(0.0225) = 0.36\,\text{J}$.

18.2 The Simple Pendulum

A simple pendulum consists of a point mass m suspended by a massless string of length ℓ.

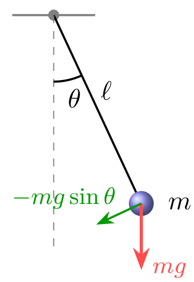


Figure 18.2.1: A simple pendulum of length ℓ. The restoring torque about the pivot is $\tau = -mg\ell\sin\theta$.

Derivation. The torque about the pivot is $\tau = -mg\ell\sin\theta$ (the minus sign indicates it is restoring). With $I = m\ell^2$:

$$m\ell^2\ddot{\theta} = -mg\ell\sin\theta \quad\Longrightarrow\quad \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.$$

This is *not* SHM because of the $\sin\theta$. But for small angles, $\sin\theta \approx \theta$ (in radians), giving:

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \quad\Longrightarrow\quad \boxed{\omega_0 = \sqrt{\frac{g}{\ell}}, \qquad T = 2\pi\sqrt{\frac{\ell}{g}}.} \tag{18.10}$$

Key Point 18.2: Amplitude Independence

The pendulum period is independent of the amplitude (for small angles) and of the mass m—a remarkable result first noted by Galileo. A heavy bob and a light bob on strings of the same length swing with the same period. This is the rotational analogue of the fact that all objects fall at the same rate in a gravitational field.

How good is the small-angle approximation? For finite amplitude θ_0, the exact period is $T = T_0[1 + \frac{1}{4}\sin^2(\theta_0/2) + \cdots]$, where $T_0 = 2\pi\sqrt{\ell/g}$ is the small-angle period. For $\theta_0 = 15°$: the error is 0.5%. For $\theta_0 = 30°$: 1.7%. For $\theta_0 = 60°$: 7%. The approximation is excellent for swings of a few degrees and reasonable up to about 20°.

Example 18.2 (Measuring g with a pendulum). A simple pendulum of length $\ell = 1.000\,\text{m}$ has period $T = 2.007\,\text{s}$. Then $g = 4\pi^2\ell/T^2 = 4\pi^2(1.000)/(2.007)^2 = 9.804\,\text{m/s}^2$. This is one of the simplest and most precise methods for measuring gravitational acceleration.

18.3 Physical Pendulums

A **physical pendulum** is any rigid body that swings about a fixed pivot under gravity.

Theorem 18.1: Physical Pendulum

A rigid body of mass M pivoted at point O, a distance d from the center of mass, oscillates (for small angles) with:

$$\boxed{\omega_0 = \sqrt{\frac{Mgd}{I_O}}, \qquad T = 2\pi\sqrt{\frac{I_O}{Mgd}},} \tag{18.11}$$

where I_O is the moment of inertia about the pivot.

Derivation. The torque about the pivot is $\tau = -Mgd\sin\theta \approx -Mgd\,\theta$ for small θ. Newton's second law for rotation: $I_O\ddot{\theta} = -Mgd\,\theta$, which is SHM with $\omega_0^2 = Mgd/I_O$.

Consistency check: for a simple pendulum ($I_O = m\ell^2$, $d = \ell$): $T = 2\pi\sqrt{m\ell^2/(mg\ell)} = 2\pi\sqrt{\ell/g}$. ✓

Example 18.3 (Uniform rod as pendulum). A uniform rod of mass M and length L pivoted at one end: $I_O = \frac{1}{3}ML^2$, $d = L/2$.

$$T = 2\pi\sqrt{\frac{ML^2/3}{Mg(L/2)}} = 2\pi\sqrt{\frac{2L}{3g}}.$$

This equals a simple pendulum of length $\ell_{\text{eff}} = 2L/3$—shorter than the rod, because the distributed mass oscillates faster than a point mass at the end.

18.4 Damped Harmonic Motion

In any real system, friction or air resistance dissipates energy and the oscillations die out. The simplest model is **viscous damping**, where the damping force is proportional to velocity: $F_{\text{damp}} = -bv$.

The equation of motion becomes:

$$m\ddot{x} + b\dot{x} + kx = 0 \quad\Longrightarrow\quad \boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0,} \tag{18.12}$$

where $\gamma = b/(2m)$ is the **damping rate** and $\omega_0 = \sqrt{k/m}$ is the natural frequency (without damping). The character of the solution depends on the relative size of γ and ω_0:

Underdamped ($\gamma < \omega_0$)

The system oscillates with exponentially decaying amplitude:

$$\boxed{x(t) = Ae^{-\gamma t}\cos(\omega_d t + \phi), \qquad \omega_d = \sqrt{\omega_0^2 - \gamma^2}.} \tag{18.13}$$

The oscillation frequency ω_d is reduced from the natural frequency, and the amplitude decays as $e^{-\gamma t}$. This is the most common case in practice.

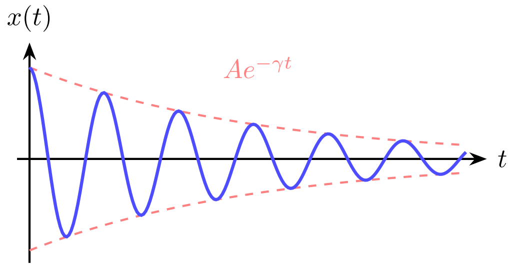


Figure 18.4.1: Underdamped oscillation: the displacement oscillates within an exponentially decaying envelope $\pm Ae^{-\gamma t}$.

Critically Damped ($\gamma = \omega_0$)

$$x(t) = (C_1 + C_2 t)\, e^{-\gamma t}. \tag{18.14}$$

The system returns to equilibrium as fast as possible *without oscillating*. This is the optimal damping for systems like door closers, shock absorbers, and galvanometers, where you want the system to settle quickly without overshooting.

Overdamped ($\gamma > \omega_0$)

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \qquad r_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} < 0. \tag{18.15}$$

The system returns to equilibrium without oscillating, but more slowly than the critically damped case. Both exponentials are decaying, but the slower one dominates at late times.

Energy Decay and the Quality Factor

For an underdamped oscillator, the energy decays as:

$$E(t) \approx E_0\, e^{-2\gamma t} = E_0\, e^{-\omega_0 t/Q}, \tag{18.16}$$

where the **quality factor** (or Q-factor) is:

$$\boxed{Q = \frac{\omega_0}{2\gamma} = \frac{\sqrt{km}}{b}.} \tag{18.17}$$

Q measures how many oscillation cycles occur before the energy decays significantly. Specifically, the energy drops to $1/e$ of its initial value in $Q/(2\pi)$ cycles. A tuning fork ($Q \sim 1000$) rings for hundreds of cycles; a critically damped door closer ($Q = 1/2$) doesn't oscillate at all.

18.5 Driven Harmonic Motion and Resonance

If an external force $F_0 \cos(\omega_{\text{ext}} t)$ drives a damped oscillator, the equation of motion becomes:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega_{\text{ext}} t). \tag{18.18}$$

Steady-State Solution

After initial transients die out (which takes a time $\sim 1/\gamma$), the system oscillates at the *driving frequency* ω_{ext} (not its natural frequency ω_0):

$$x(t) = A(\omega_{\text{ext}}) \cos(\omega_{\text{ext}} t - \delta), \tag{18.19}$$

where the amplitude and phase lag are:

$$\boxed{A(\omega_{\text{ext}}) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_{\text{ext}}^2)^2 + (2\gamma\omega_{\text{ext}})^2}},} \tag{18.20}$$

$$\tan\delta = \frac{2\gamma\omega_{\text{ext}}}{\omega_0^2 - \omega_{\text{ext}}^2}. \tag{18.21}$$

Resonance

The amplitude $A(\omega_{\text{ext}})$ is maximized near $\omega_{\text{ext}} \approx \omega_0$ (for light damping, exactly at ω_0). This is **resonance**: the driving force pumps energy into the system most efficiently when it drives at the natural frequency.

At resonance ($\omega_{\text{ext}} = \omega_0$):

$$A_{\text{res}} = \frac{F_0}{2m\gamma\omega_0} = \frac{F_0}{k} \cdot Q. \tag{18.22}$$

The resonant amplitude is Q times the static displacement F_0/k. The width (FWHM) of the resonance peak is:

$$\Delta\omega \approx \frac{\omega_0}{Q} = 2\gamma. \tag{18.23}$$

A high-Q system has a sharp, tall resonance peak; a low-Q system has a broad, low one.

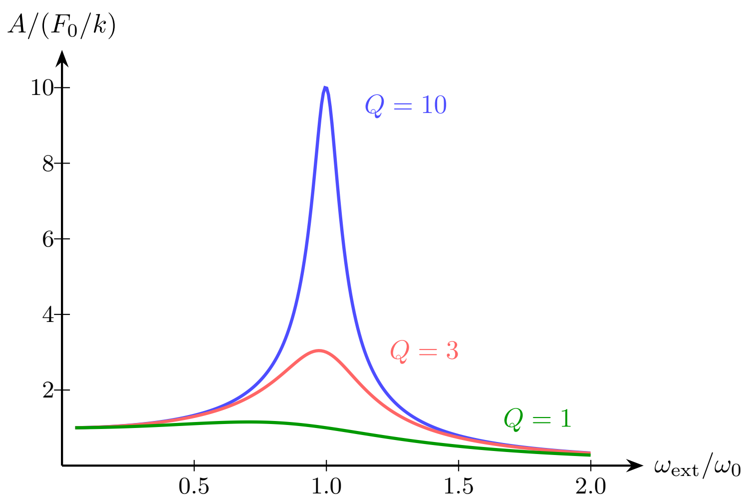


Figure 18.5.1: Resonance curves for a driven damped oscillator at three different quality factors. Higher Q gives a taller, narrower peak near $\omega_{\text{ext}} = \omega_0$.

The **phase lag** δ varies from 0 (driving well below resonance—the response is in phase with the force) through $\pi/2$ (at resonance) to π (well above resonance—the response is opposite to the force).

Resonance Without Damping

For an undamped oscillator driven exactly at ω_0, the amplitude grows linearly in time without bound:

$$x_p(t) = \frac{F_0}{2m\omega_0}\, t \sin(\omega_0 t). \tag{18.24}$$

In reality, damping (however small) always limits the growth, but this formula explains why soldiers break step on a bridge and why opera singers can shatter wine glasses.

Example 18.4 (Resonance catastrophe). The Tacoma Narrows Bridge collapse (1940) is often attributed to resonance: the wind drove oscillations near the bridge's natural torsional frequency. While the full explanation involves aerodynamic flutter (a self-excited oscillation), the basic principle illustrates why engineers must ensure that ω_0 of a structure does not coincide with common driving frequencies.

18.6 Connections Between Oscillatory Systems

The SHM equation $\ddot{x} + \omega_0^2 x = 0$ appears in a remarkable variety of physical systems. The key insight is that *any* restoring force linear in displacement produces SHM:

System	Variable	ω_0	Condition
Mass on spring	x	$\sqrt{k/m}$	Hooke's law
Simple pendulum	θ	$\sqrt{g/\ell}$	Small angles
Physical pendulum	θ	$\sqrt{Mgd/I_O}$	Small angles
Torsional pendulum	θ	$\sqrt{\kappa/I}$	Linear restoring torque
Floating object	x	$\sqrt{\rho g A/M}$	Linear buoyancy
Tunnel through Earth	r	$\sqrt{g_s/R}$	Uniform density
LC circuit	q	$\sqrt{1/(LC)}$	No resistance

Example 18.5 (Springs in parallel and series). Two springs with constants k_1 and k_2 attached to a mass m. In *parallel*: $k_{\text{eff}} = k_1 + k_2$ (forces add), $\omega = \sqrt{(k_1+k_2)/m}$. In *series*: $1/k_{\text{eff}} = 1/k_1 + 1/k_2$, so $k_{\text{eff}} = k_1 k_2/(k_1+k_2)$.

Example 18.6 (Vertical spring). A mass m hangs from a spring (k) in gravity. Equilibrium extension: $x_0 = mg/k$. Let $\xi = x - x_0$. Then $m\ddot{\xi} = -k\xi$ (gravity and the equilibrium spring force cancel exactly), so $\omega = \sqrt{k/m}$—*identical* to the horizontal case. Gravity shifts the equilibrium but does not change the frequency.

Example 18.7 (Q-factor and energy decay). An underdamped oscillator with $Q = 50$. (a) Energy drops to $1/e$ in $Q/(2\pi) \approx 8$ cycles. (b) Amplitude drops to half in $Q \ln 2/\pi \approx 11$ cycles.

18.7 Superposition and Beats

When two SHM oscillations of equal amplitude but slightly different frequency are superposed:

$$x(t) = A\cos(\omega_1 t) + A\cos(\omega_2 t) = 2A\cos\left(\frac{\omega_1-\omega_2}{2}t\right)\cos\left(\frac{\omega_1+\omega_2}{2}t\right). \tag{18.25}$$

The result is a rapid oscillation at the average frequency $\bar{\omega} = (\omega_1+\omega_2)/2$, modulated by a slow "envelope" at the **beat frequency**:

$$f_{\text{beat}} = |f_1 - f_2|. \tag{18.26}$$

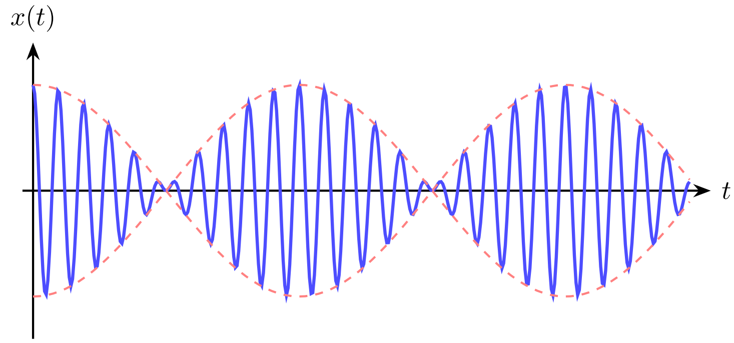


Figure 18.7.1: Beats: two oscillations of slightly different frequency produce an amplitude modulation. The envelope (dashed) oscillates at the beat frequency $f_{\text{beat}} = |f_1 - f_2|$.

Beats are heard when two tuning forks of slightly different pitch are struck simultaneously: the sound periodically swells and fades at the beat frequency. Musicians use beats to tune instruments—when the beat frequency drops to zero, the two notes are in unison.

Problems

Problem 18.1 ⋆⋆

A block of mass $m = 0.50\,\text{kg}$ attached to a spring ($k = 32\,\text{N/m}$) is displaced $A = 0.15\,\text{m}$ from equilibrium and released from rest. (a) Find ω_0, T, and f. (b) Show that $x(t) = A\cos(\omega_0 t)$ satisfies the equation of motion. (c) Find the maximum speed and where it occurs. (d) Compute the total energy and verify it is conserved.

Problem 18.2 ⋆⋆⋆

A solid cylinder (mass M, cross-sectional area A) floats upright in a liquid of density ρ. (a) Find the equilibrium depth of submersion d_0. (b) Show that if the cylinder is pushed down a small distance x and released, it undergoes SHM. Find ω. (c) Show that $T = 2\pi\sqrt{d_0/g}$. (d) Compute for $M = 2.0\,\text{kg}$, $A = 8.0 \times 10^{-3}\,\text{m}^2$, $\rho = 1000\,\text{kg/m}^3$.

Problem 18.3 ⋆⋆⋆

A uniform rod (mass M, length L) is pivoted at one end. A small disk (mass m, radius $r \ll L$) is attached at the other end. (a) Find the moment of inertia I about the pivot. (b) Find the distance d from the pivot to the combined CM. (c) Find the period T for small oscillations. (d) Recover $T = 2\pi\sqrt{2L/(3g)}$ as $m \to 0$. (e) Compute for $M = 0.50\,\text{kg}$, $L = 0.80\,\text{m}$, $m = 0.30\,\text{kg}$, $r = 0.05\,\text{m}$.

Problem 18.4 ⋆⋆⋆

A damped oscillator has $m = 0.40\,\text{kg}$, $k = 20\,\text{N/m}$, $b = 0.50\,\text{kg/s}$. (a) Find γ, ω_0, and ω_d. Is the system underdamped, critically damped, or overdamped? (b) Find the energy half-life (time for E to drop to $E_0/2$). (c) Find the Q-factor and the maximum steady-state amplitude if driven at resonance with $F_0 = 2.0\,\text{N}$.

Problem 18.5 ⋆⋆⋆⋆⋆

A torsional pendulum consists of a uniform disk (mass M, radius R) suspended by a wire that provides a restoring torque $\tau = -\kappa\theta$ and angular damping $\tau_{\text{damp}} = -\beta\dot{\theta}$. (a) Write the equation of motion and identify ω_0 and γ. (b) For $M = 0.80\,\text{kg}$, $R = 0.10\,\text{m}$, $\kappa = 0.50\,\text{N m/rad}$, $\beta = 0.010\,\text{N m s/rad}$: is the system underdamped, critically damped, or overdamped? (c) Find ω_d and Q. (d) How many cycles elapse before the amplitude decays to $1/e$ of its initial value?

Problem 18.6 ⋆⋆⋆

A block of mass m is connected to two identical springs (k each), one on each side, on a frictionless surface. Both springs are at natural length at equilibrium. Show that for longitudinal displacements, the angular frequency is $\omega = \sqrt{2k/m}$ and find the period.

Problem 18.7 ⋆⋆⋆⋆

A tunnel is drilled through the Earth (not through the center) connecting two cities separated by angle 2α as seen from the center. A train is released from rest at one end. (a) Show that the gravitational force component along the tunnel is proportional to displacement from the midpoint. (b) Show that the travel time is $T/2 = \pi\sqrt{R/g_s} \approx 42\,\text{min}$, independent of α. (c) Compare to the diametral tunnel travel time from Chapter 15.

Problem 18.8 ⋆⋆⋆⋆

A mass m hangs from a spring of constant k in a gravitational field g. (a) Find the equilibrium extension x_0. (b) Show that oscillations about this equilibrium have frequency $\omega = \sqrt{k/m}$, identical to the horizontal case. (c) Using $\xi = x - x_0$, show that $E = \frac{1}{2}m\dot{\xi}^2 + \frac{1}{2}k\xi^2$.

Problem 18.9 ⋆⋆

A mass $m = 2.0\,\mathrm{kg}$ on a spring ($k = 50\,\mathrm{N/m}$) oscillates with $A = 0.10\,\mathrm{m}$. (a) Find the period. (b) Find the maximum speed. (c) Find the maximum acceleration. (d) Find the speed when $x = A/2$.

Problem 18.10 ⋆⋆⋆

Two springs ($k_1 = 100\,\mathrm{N/m}$, $k_2 = 200\,\mathrm{N/m}$) are connected in series to a $0.50\,\mathrm{kg}$ mass. Find the period. Repeat for parallel connection.

Problem 18.11 ⋆⋆⋆⋆

A uniform solid cylinder (mass M, radius R) rolls without slipping inside a larger fixed cylinder of radius $\mathcal{R} > R$. (a) Show that for small angular displacements from the bottom, the motion is SHM. (b) Find $\omega = \sqrt{2g/[3(\mathcal{R} - R)]}$. (c) Compare to a sliding (frictionless) point mass inside the large cylinder and explain the difference.

Part II

Solutions

Solutions to All Problems

Below are complete solutions to every problem in this textbook. The solution number matches the problem number (Chapter.Problem).

Chapter 1: The Nature of Physics

Solution 1.1

(a) $[E] = \mathsf{ML^2T^{-2}}$. (b) $[P] = [E/t] = \mathsf{ML^2T^{-3}}$. (c) $[p] = [F/A] = \mathsf{ML^{-1}T^{-2}}$. (d) From $F = Gm_1m_2/r^2$: $[G] = [Fr^2/(m^2)] = \mathsf{M^{-1}L^3T^{-2}}$. (e) From $F = kx$: $[k] = [F/x] = \mathsf{MT^{-2}}$.

Solution 1.2

Let $F \propto R^a v^b \rho^c$. Then $\mathsf{MLT^{-2}} = \mathsf{L}^a(\mathsf{LT^{-1}})^b(\mathsf{ML^{-3}})^c = \mathsf{M}^c\mathsf{L}^{a+b-3c}\mathsf{T}^{-b}$. From M: $c = 1$. From T: $b = 2$. From L: $a + 2 - 3 = 1$, so $a = 2$. Therefore $F \propto \rho R^2 v^2$.

Solution 1.3

$[v^2] = \mathsf{L^2T^{-2}}$ and $[2ax^3] = (\mathsf{LT^{-2}})(\mathsf{L^3}) = \mathsf{L^4T^{-2}}$. Since $\mathsf{L^2T^{-2}} \neq \mathsf{L^4T^{-2}}$, the equation is dimensionally inconsistent and must be wrong.

Solution 1.4

Let $v \propto T^a\mu^b$. Then $\mathsf{LT^{-1}} = (\mathsf{MLT^{-2}})^a(\mathsf{ML^{-1}})^b = \mathsf{M}^{a+b}\mathsf{L}^{a-b}\mathsf{T}^{-2a}$. From T: $-2a = -1$, $a = 1/2$. From M: $b = -1/2$. Check L: $1/2 + 1/2 = 1$. ✓

So $v = C\sqrt{T/\mu}$.

Solution 1.5

Assumptions: 3 million people, average household ~ 3 people (10^6 households), $\sim 5\%$ own pianos (5×10^4 pianos), each tuned once/year. A tuner does ~ 4 tunings/day, works ~ 250 days/year (~ 1000/year). Number of tuners $\approx 5 \times 10^4/10^3 = 50$. (Order of magnitude: ~ 30–100.)

Solution 1.6

$[T] = [\sqrt{m/k}]$. $[m/k] = \mathsf{M}/(\mathsf{MT^{-2}}) = \mathsf{T^2}$. So $[\sqrt{m/k}] = \mathsf{T}$. The 2π is dimensionless. ✓

Solution 1.7

$[a_c] = \mathsf{LT^{-2}}$. Let $a_c \propto r^aT^b$. Then $\mathsf{LT^{-2}} = \mathsf{L}^a\mathsf{T}^b$. From L: $a = 1$. From T: $b = -2$. So $a_c \propto r/T^2$ (i.e., $a_c = 4\pi^2r/T^2$).

Solution 1.8

(a) $100 \text{ km/hr} \times \frac{1000 \text{ m}}{1 \text{ km}} \times \frac{1 \text{ hr}}{3600 \text{ s}} = 27.8 \text{ m/s}$.
(b) $1 \text{ yr} = 365.25 \times 24 \times 3600 = 3.156 \times 10^7$ s.
(c) $1 \text{ ly} = c \times 1 \text{ yr} = 3 \times 10^8 \times 3.156 \times 10^7 = 9.47 \times 10^{15}$ m.

Solution 1.9

$[F_{\text{rad}}] = \mathsf{MLT^{-2}}$. Let $F \propto L^ac^br^d$. $[L] = \mathsf{ML^2T^{-3}}$, $[c] = \mathsf{LT^{-1}}$, $[r] = \mathsf{L}$. From M: $a = 1$. From T: $-3a - b = -2$, $b = -1$. From L: $2a + b + d = 1$, $d = -2$. So $F \propto L/(cr^2)$.

Solution 1.10

Room $\sim 5\times5\times3 = 75$ m^3. Air density ~ 1.2 kg/m^3, so $M \sim 90$ kg. Molar mass ~ 0.029 kg/mol; molecules $\sim 90/0.029 \times 6\times10^{23} \sim 1.9\times10^{27}$. At room temperature ($T \sim 300$ K), $\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}k_BT$. Total KE $= \frac{3}{2}Nk_BT = \frac{3}{2}(n)(RT) = \frac{3}{2}\frac{M}{\mu}RT = \frac{3}{2}\frac{90}{0.029}(8.314)(300) \approx 1.2\times10^7$ J ~ 10 MJ.

Chapter 2: Vectors and Coordinate Systems

Solution 2.1

(a) $\boldsymbol{A}+\boldsymbol{B} = 2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$.
(b) $|\boldsymbol{A}| = \sqrt{16+9+4} = \sqrt{29}$; $|\boldsymbol{B}| = \sqrt{4+25+1} = \sqrt{30}$.
(c) $\boldsymbol{A}\cdot\boldsymbol{B} = -8-15-2 = -25$.
(d) $\cos\theta = -25/\sqrt{870}$, $\theta \approx 147.9°$.

Solution 2.2

(a) $\boldsymbol{C} = 10\cos120°\,\hat{\boldsymbol{\imath}} + 10\sin120°\,\hat{\boldsymbol{\jmath}} = -5\hat{\boldsymbol{\imath}} + 5\sqrt{3}\hat{\boldsymbol{\jmath}}$.
(b) $\hat{\boldsymbol{C}} = -\frac{1}{2}\hat{\boldsymbol{\imath}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}}$.
(c) Rotate 90°: $\boldsymbol{D} = -5\sqrt{3}\hat{\boldsymbol{\imath}} - 5\hat{\boldsymbol{\jmath}}$ (or the negative).

Solution 2.3

(a) $\boldsymbol{u}\cdot\boldsymbol{v} = 2-2\alpha-3 = 0 \Rightarrow \alpha = -1/2$.
(b) $\boldsymbol{w}\cdot\boldsymbol{v} = 4-2+\beta = 0 \Rightarrow \beta = -2$.
(c) $\boldsymbol{u}\cdot\boldsymbol{w} = 8-0.5+6 = 13.5 \neq 0$. Not perpendicular. Perpendicularity is not transitive: vectors perpendicular to $\boldsymbol{v}$ form a plane, and two vectors in that plane need not be perpendicular to each other.

Solution 2.4

$|\boldsymbol{A}\times\boldsymbol{B}|^2 + (\boldsymbol{A}\cdot\boldsymbol{B})^2 = |\boldsymbol{A}|^2|\boldsymbol{B}|^2(\sin^2\theta+\cos^2\theta) = |\boldsymbol{A}|^2|\boldsymbol{B}|^2$.
Verification: $\boldsymbol{A}\cdot\boldsymbol{B} = 3-8 = -5$; $\boldsymbol{A}\times\boldsymbol{B} = -8\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}} - 6\hat{\boldsymbol{k}}$. LHS: $200+25 = 225$. RHS: $9\times25 = 225$. ✓

Solution 2.5

(a) $\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{r}|^2 = 2\boldsymbol{r}\cdot\boldsymbol{v} = 0$, so $|\boldsymbol{r}| =$ const.
(b) The particle moves on a sphere (circle in 2D) of fixed radius centered at the origin.
(c) $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{v}$ is perpendicular to both $\boldsymbol{r}$ and $\boldsymbol{v}$. Since $\boldsymbol{L}$ is constant, $\boldsymbol{r}$ always lies in a fixed plane perpendicular to $\boldsymbol{L}$.

Solution 2.6

Unit vectors: $\hat{\boldsymbol{A}} = (3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}})/5$, $\hat{\boldsymbol{B}} = (4\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}})/5$. Bisector direction: $\hat{\boldsymbol{A}} + \hat{\boldsymbol{B}} = (7\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})/5$. Magnitude $= \sqrt{49+1}/5 = \sqrt{50}/5$. Unit vector $= (7\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})/\sqrt{50} = (7\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})/(5\sqrt{2})$.

Solution 2.7

(a) $\boldsymbol{v} = 6t\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 6t^2\hat{\boldsymbol{k}}$; at $t=1$: $\boldsymbol{v} = 6\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}$. $\boldsymbol{a} = 6\hat{\boldsymbol{\imath}} + 12t\hat{\boldsymbol{k}}$; at $t=1$: $\boldsymbol{a} = 6\hat{\boldsymbol{\imath}} + 12\hat{\boldsymbol{k}}$.
(b) $|\boldsymbol{v}| = \sqrt{36+16+36} = \sqrt{88} = 2\sqrt{22} \approx 9.38$ m/s.
(c) $\cos\alpha = \boldsymbol{v}\cdot\boldsymbol{a}/(|\boldsymbol{v}||\boldsymbol{a}|) = (36+0+72)/(\sqrt{88}\sqrt{180}) = 108/\sqrt{15840}$. $\alpha \approx 30.8°$.

Solution 2.8

$\boldsymbol{B}\times\boldsymbol{C} = (\hat{\imath}+\hat{\jmath})\times(\hat{\imath}+\hat{\jmath}+\hat{k}) = \hat{\imath}\times\hat{k}+\hat{\jmath}\times\hat{\imath}+\hat{\jmath}\times\hat{k} = -\hat{\jmath}-\hat{k}+\hat{\imath} = \hat{\imath}-\hat{\jmath}-\hat{k}$. $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \hat{\imath}\cdot(\hat{\imath}-\hat{\jmath}-\hat{k}) = 1$. Volume $= |1| = 1$ cubic unit.

Solution 2.9

$|\boldsymbol{A}+\boldsymbol{B}|^2 = (\boldsymbol{A}+\boldsymbol{B})\cdot(\boldsymbol{A}+\boldsymbol{B}) = A^2 + 2\boldsymbol{A}\cdot\boldsymbol{B} + B^2$. Similarly $|\boldsymbol{A}-\boldsymbol{B}|^2 = A^2 - 2\boldsymbol{A}\cdot\boldsymbol{B} + B^2$. Sum $= 2A^2 + 2B^2$. Geometrically: the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of all four sides.

Solution 2.10

$F_\parallel = \boldsymbol{F}\cdot\hat{\boldsymbol{n}} = 5(3/5) + 12(4/5) = 3 + 9.6 = 12.6$ N. $\boldsymbol{F}_\parallel = 12.6\hat{\boldsymbol{n}} = 7.56\hat{\imath} + 10.08\hat{\jmath}$. $\boldsymbol{F}_\perp = \boldsymbol{F} - \boldsymbol{F}_\parallel = -2.56\hat{\imath} + 1.92\hat{\jmath}$. Magnitude: $|\boldsymbol{F}_\perp| = \sqrt{6.55 + 3.69} = 3.2$ N.

Chapter 3: Kinematics in One Dimension

Solution 3.1

(a) $a = 25/8 = 3.125\,\text{m/s}^2$. (b) $x = \frac{1}{2}(3.125)(64) = 100\,\text{m}$.

Solution 3.2

Ball 1 (dropped): $y_1 = H - \frac{1}{2}gt^2$. Ball 2 (up): $y_2 = v_0t - \frac{1}{2}gt^2$.
(a) $y_1 = y_2 \Rightarrow H = v_0t \Rightarrow t = H/v_0$. Height: $y = H - gH^2/(2v_0^2)$.
(b) $v_1 = -gH/v_0$ (down); $v_2 = (v_0^2 - gH)/v_0$.
(c) $y = H/2$ gives $gH^2/(2v_0^2) = H/2$, so $v_0 = \sqrt{gH}$.

Solution 3.3

(a) $x_A = 1.5t^2$; $x_B = 20(t-2)$ for $t \geq 2$.
(b) $1.5t^2 = 20(t-2) \Rightarrow 3t^2 - 40t + 80 = 0 \Rightarrow t \approx 2.5$ s and 10.9 s.
(c) $3t = 20 \Rightarrow t = 6.67\,\text{s}$.
(d) At $t = 6.67$ s: $x_A = 66.7$ m, $x_B = 93.3$ m, $\Delta x_{\max} = 26.7\,\text{m}$.

Solution 3.4

(a) $a = -2\beta t$; $x = v_0t - \beta t^3/3$.
(b) $x = 0$: $t(v_0 - \beta t^2/3) = 0 \Rightarrow t = \sqrt{3v_0/\beta}$.
(c) Reversal at $t_r = \sqrt{v_0/\beta}$; $x(t_r) = \frac{2v_0}{3}\sqrt{v_0/\beta}$. Total distance $= 2x(t_r) = \frac{4v_0}{3}\sqrt{v_0/\beta}$.

Solution 3.5

(a) $v = a_0t - kt^2/2$; $x = a_0t^2/2 - kt^3/6$.
(b) At $t_1 = a_0/k$: $v_1 = a_0^2/(2k)$, $x_1 = a_0^3/(3k^2)$. After cutoff: $h_{\max} = x_1 + v_1^2/(2g) = a_0^3/(3k^2) + a_0^4/(8gk^2)$.
(c) $h_{\max} = 27000/9 \cdot (1/3 + 30/80) = 3000 \times 17/24 = 2125\,\text{m}$.

Solution 3.6

(a) $v^2 = v_0^2 - 2gh \Rightarrow h = v_0^2/(2g) = 400/20 = 20$ m. (b) Total time $= 2v_0/g = 40/10 = 4.0$ s. (c) By symmetry, $v = 20$ m/s (downward).

Solution 3.7

$v(t) = 2 + \int_0^t 6t'\,\mathrm{d}t' = 2 + 3t^2$. $x(t) = 1 + \int_0^t (2 + 3t'^2)\mathrm{d}t' = 1 + 2t + t^3$. At $t = 2$: $v = 14$ m/s, $x = 1 + 4 + 8 = 13$ m.

Solution 3.8

Each train stops in time $t_s = 30/3 = 10$ s, traveling $d_s = 30(10) - \frac{1}{2}(3)(100) = 150$ m. Total braking distance $= 2(150) = 300$ m < 500 m. No collision.

Solution 3.9

(a) Police: $x_p = \frac{1}{2}(3)t^2$. Speeder: $x_s = 30t$. Equal when $\frac{3}{2}t^2 = 30t$, so $t = 20$ s. (b) $v_p = 3(20) = 60$ m/s.

Solution 3.10

At $t = 2$ s, elevator speed $= 2(2) = 4$ m/s, floor position $= \frac{1}{2}(2)(4) = 4$ m. Bolt starts at floor $+3 = 7$ m with $v_0 = 4$ m/s (upward). In elevator frame, bolt falls 3 m from rest with effective $g_{\text{eff}} = g + a_{\text{elev}} = 11.8$ m/s^2. Time: $3 = \frac{1}{2}(11.8)t^2$, $t = 0.71$ s.

Chapter 4: Kinematics in Two and Three Dimensions

Solution 4.1

(a) $80 = \frac{1}{2}(10)t^2 \Rightarrow t = 4.0\,\text{s}$. (b) $x = 15 \times 4 = 60\,\text{m}$. (c) $v_y = 40$ m/s; $v = \sqrt{15^2 + 40^2} = 42.7\,\text{m/s}$.

Solution 4.2

(a) $T = 2(40)\sin 60°/10 = 6.93\,\text{s}$. $H = 1600(0.75)/20 = 60\,\text{m}$. $R = 1600 \sin 120°/10 = 138.6\,\text{m}$.
(b) 30°. (c) At 30°: shorter flight time, lower max height, same range.

Solution 4.3

(a) $\dot{\theta} = \omega_0 + \alpha t$; $\ddot{\theta} = \alpha$.
(b) $|a_c| = R(\omega_0 + \alpha t)^2$; $|a_t| = R\alpha$.
(c) $(\omega_0 + \alpha t)^2 = \alpha \Rightarrow t = (\sqrt{\alpha} - \omega_0)/\alpha$ (if $\sqrt{\alpha} > \omega_0$).

Solution 4.4

(a) Time: $t = \frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gh}}{g}$. $R = (v_0 \cos\theta)t$. $v_f = \sqrt{v_0^2 + 2gh}$.
(b) v_f has no θ dependence (energy conservation).
(c) Optimal angle: $\theta_{\text{opt}} = \arctan\left(\frac{v_0}{\sqrt{v_0^2 + 2gh}}\right)$.

Solution 4.5

(a) $x^2/A^2 + y^2/B^2 = \cos^2\omega t + \sin^2\omega t = 1$. ✓
(b) $\boldsymbol{v} = -A\omega \sin\omega t\,\hat{\boldsymbol{\imath}} + B\omega\cos\omega t\,\hat{\boldsymbol{\jmath}}$; $\boldsymbol{a} = -A\omega^2\cos\omega t\,\hat{\boldsymbol{\imath}} - B\omega^2 \sin\omega t\,\hat{\boldsymbol{\jmath}}$.
(c) $\boldsymbol{a} = -\omega^2(A\cos\omega t\,\hat{\boldsymbol{\imath}} + B\sin\omega t\,\hat{\boldsymbol{\jmath}}) = -\omega^2\boldsymbol{r}$. ✓
(d) $v = \omega\sqrt{A^2\sin^2\omega t + B^2\cos^2\omega t}$. Max $= A\omega$ at $(0, \pm B)$; min $= B\omega$ at $(\pm A, 0)$.

Solution 4.6

$\omega = 12000\times 2\pi/60 = 400\pi$ rad/s. $a_c = R\omega^2 = 0.15(400\pi)^2 = 0.15\times 1.58\times 10^6 \approx 2.37\times 10^5$ m/s^2 $\approx$ 24,200g. Force: $F = ma_c = 0.010 \times 2.37 \times 10^5 = 2370$ N.

Solution 4.7

Position: $\boldsymbol{r} = a\theta\,\hat{\boldsymbol{r}}$. With $\dot\theta = \omega_0$: $\dot r = a\omega_0$, $\ddot r = 0$.
$\boldsymbol{v} = a\omega_0\,\hat{\boldsymbol{r}} + a\theta\omega_0\,\hat{\boldsymbol{\theta}}$.
$\boldsymbol{a} = (0 - a\theta\omega_0^2)\hat{\boldsymbol{r}} + (0 + 2a\omega_0^2)\hat{\boldsymbol{\theta}} = -a\omega_0^2\theta\,\hat{\boldsymbol{r}} + 2a\omega_0^2\,\hat{\boldsymbol{\theta}}$.

Solution 4.8

Projectile: $v_x = 50\cos 30° = 43.3$ m/s, $v_{y0} = 50\sin 30° = 25$ m/s. Ground level at $y = 0$; launch at $y = 100$ m.

(a) $0 = 100 + 25t - 5t^2$, i.e., $t^2 - 5t - 20 = 0$. $t = (5+\sqrt{105})/2 = 7.62$ s. Range: $R = 43.3\times 7.62 = 330$ m.

(b) By energy: $v = \sqrt{v_0^2 + 2gh} = \sqrt{2500 + 2000} = 67.1$ m/s.

(c) $v_y = 25 - 10(7.62) = -51.2$ m/s. Impact angle: $\alpha = \arctan(51.2/43.3) = 49.8°$ below horizontal.

Solution 4.9

$\omega(1) = 3(1)^2 = 3$ rad/s. $\alpha = \mathrm{d}\omega/\mathrm{d}t = 6t$; at $t = 1$: $\alpha = 6$ rad/s^2.

(a) $v = R\omega = 2(3) = 6$ m/s. (b) $a_c = R\omega^2 = 2(9) = 18$ m/s^2. (c) $a_t = R\alpha = 2(6) = 12$ m/s^2. (d) $a = \sqrt{18^2 + 12^2} = \sqrt{468} = 21.6$ m/s^2 at $\arctan(12/18) = 33.7°$ from the radius.

Solution 4.10

The trajectory for launch angle θ is $y = x\tan\theta - \frac{gx^2}{2v_0^2}\sec^2\theta$. Let $u = \tan\theta$: $y = xu - \frac{gx^2}{2v_0^2}(1+u^2)$.

The envelope is found by $\partial y/\partial u = 0$: $x - gx^2u/v_0^2 = 0$, giving $u = v_0^2/(gx)$. Substituting back: $y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$. This is a downward parabola with vertex at $(0, v_0^2/(2g))$. Any point above this curve satisfies $y > v_0^2/(2g) - gx^2/(2v_0^2)$ and is unreachable by any projectile launched at speed v_0.

Chapter 5: Newton's Laws of Motion

Solution 5.1

(a) $N = mg = 50$ N; $f_k = \mu_k N = 0.2 \times 50 = 10$ N.
(b) $a = (30 - 10)/5 = 4.0$ m/s^2.
(c) $v = at = 4 \times 4 = 16$ m/s.

Solution 5.2

(a) $a = F/(m_1 + m_2) = 24/8 = 3.0$ m/s^2.
(b) $T = m_1 a = 3 \times 3 = 9.0$ N.
(c) $a_{\max} = 15/3 = 5$ m/s^2; $F_{\max} = 8 \times 5 = 40$ N.

Solution 5.3

(a) At the point of slip: $mg\sin\theta = \mu_s mg\cos\theta \Rightarrow \theta_{\max} = \arctan\mu_s$.

(b) For $\theta < \theta_{\max}$: the block is stationary, so $f = mg\sin\theta$ (static friction matches the gravitational component).

(c) For $\theta > \theta_{\max}$: $ma = mg\sin\theta - \mu_k mg\cos\theta$, so $a = g(\sin\theta - \mu_k\cos\theta)$.

Solution 5.4

Let the half-angle from vertical be φ, where $\sin\varphi = R/L$, $\cos\varphi = \sqrt{L^2 - R^2}/L$.
(a) Vertical: $T\cos\varphi = mg \Rightarrow T = mgL/\sqrt{L^2 - R^2}$.
(b) Horizontal: $T\sin\varphi = mv^2/R \Rightarrow v = R\sqrt{g/\sqrt{L^2 - R^2}}$.
(c) $\tau = 2\pi R/v = 2\pi\sqrt[4]{(L^2 - R^2)}/\sqrt{g}$. More precisely, $\tau = 2\pi\sqrt{\sqrt{L^2 - R^2}/g}$.
(d) As $R \to 0$: $\sqrt{L^2 - R^2} \to L$, so $\tau \to 2\pi\sqrt{L/g}$. ✓

Solution 5.5

(a) $m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - bv$.
(b) This is a first-order linear ODE. Separating: $\frac{\mathrm{d}v}{g - bv/m} = \mathrm{d}t$. Integrating with $v(0) = 0$: $v(t) = v_T(1 - e^{-bt/m})$ where $v_T = mg/b$.
(c) Terminal velocity: $v_T = mg/b$. Position: $y(t) = v_T t - v_T\tau(1 - e^{-t/\tau})$ where $\tau = m/b$.
(d) For small t: $e^{-t/\tau} \approx 1 - t/\tau + t^2/(2\tau^2)$. $y \approx v_T t - v_T\tau(t/\tau - t^2/(2\tau^2)) = v_T t^2/(2\tau) = \frac{1}{2}gt^2$. ✓

Solution 5.6

(a) At the crest, the forces on the car are: weight mg downward and normal force N upward.
(b) The centripetal direction is downward (toward center of curvature): $mg - N = mv^2/R$, so $N = m(g - v^2/R)$.
(c) Contact lost when $N = 0$: $v = \sqrt{gR}$.

Solution 5.7

(a) For mass m_1 (upward positive): $T - m_1 g = m_1 a$. For mass m_2 (downward positive): $m_2 g - T = m_2 a$. Adding: $(m_2 - m_1)g = (m_1 + m_2)a$, so $a = \frac{(m_2 - m_1)g}{m_1 + m_2}$.
(b) $T = m_1(g + a) = \frac{2m_1 m_2 g}{m_1 + m_2}$.
(c) If $m_1 = m_2$: $a = 0$ and $T = mg$. ✓
(d) If $m_1 \ll m_2$: $a \to g$ and $T \to 2m_1 g$. ✓

Solution 5.8

The wall pushes horizontally with normal force $N = F$. Friction (vertical, upward) must support the weight: $f_s = \mu_s N = \mu_s F \geq mg$. Therefore $F_{\min} = mg/\mu_s$.

Solution 5.9

(a) At angle θ from the bottom, the radial equation is $mg\cos\theta - N = mv^2/R$. Energy conservation from release at angle θ_0 (measured from bottom): $v^2 = 2gR(\cos\theta - \cos\theta_0)$. Substituting: $N = m[3g\cos\theta - 2g\cos\theta_0]$.
(b) The bead leaves the wire when $N = 0$: $\cos\theta^* = \frac{2}{3}\cos\theta_0$.

Solution 5.10

All three blocks accelerate together: $a = F/(m_1 + m_2 + m_3)$. String between m_1 and m_2: $T_1 = m_1 a = m_1 F/(m_1 + m_2 + m_3)$. String between m_2 and m_3: $T_2 = (m_1 + m_2)a = (m_1 + m_2)F/(m_1 + m_2 + m_3)$.

Chapter 6: Work and Power

Solution 6.1

$W = Fd = 40 \times 5.0 = 200\,\text{J}$. By $W = \frac{1}{2}mv^2$: $v = \sqrt{2W/m} = \sqrt{400/10} = 6.32\,\text{m/s}$.

Solution 6.2

$W = \int_1^4 (6x^2 - 2x)\,\mathrm{d}x = [2x^3 - x^2]_1^4 = (128 - 16) - (2 - 1) = 111\,\text{J}$.

Solution 6.3

(a) Along the incline (x-axis up, y perpendicular): $W_F = Fd$. $W_g = -mgd\sin\theta$. $W_N = 0$ (perpendicular). $W_f = -\mu_k mgd\cos\theta$.

(b) $W_{\text{net}} = Fd - mgd(\sin\theta + \mu_k\cos\theta) = \frac{1}{2}mv^2$, so $v = \sqrt{\frac{2d}{m}[F - mg(\sin\theta + \mu_k\cos\theta)]}$.

(c) For $v^2 > 0$: $F > mg(\sin\theta + \mu_k\cos\theta)$.

Solution 6.4

(a) The block stops permanently when the spring force at the turning point cannot overcome static friction: $k|x| \le \mu_k mg$, giving $|x| \le \mu_k mg/k$.

(b) From start ($x = A$, $K = 0$) to final rest ($|x_f| \le \mu_k mg/k$, $K = 0$): $\frac{1}{2}kA^2 - \frac{1}{2}kx_f^2 = \mu_k mg\, d_{\text{total}}$. Since $|x_f| \le \mu_k mg/k$ and typically $A \gg \mu_k mg/k$: $d_{\text{total}} \approx kA^2/(2\mu_k mg)$.

(c) $d = 200(0.04)/(2 \times 0.15 \times 0.50 \times 10) = 8/1.5 \approx 5.33\,\text{m}$.

Solution 6.5

(a) $v_{\max} = P/F_{\text{resist}} = 60000/600 = 100\,\text{m/s}$.

(b) $F_{\text{engine}} = P/v = 60000/40 = 1500\,\text{N}$. $F_{\text{net}} = 1500 - 600 = 900\,\text{N}$. $a = 900/1200 = 0.75\,\text{m/s}^2$.

Solution 6.6

(a) At terminal speed: $P = bv_T^3$, so $v_T = (P/b)^{1/3}$.

(b) Net force: $F_{\text{net}} = P/v - bv^2$. Newton's second law: $m\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{P}{v} - bv^2$.

(c) As $v \to v_T^-$: $\frac{\mathrm{d}v}{\mathrm{d}t} \to 0$. Near v_T, set $v = v_T - \epsilon$: the denominator $P - bv^3 \approx 3bv_T^2\epsilon$, so $\mathrm{d}t \sim \mathrm{d}\epsilon/\epsilon$, which integrates to $\ln\epsilon$ and diverges as $\epsilon \to 0$. The time is formally infinite.

Solution 6.7

$W = \boldsymbol{F} \cdot \boldsymbol{d} = (3)(2) + (4)(-1) = 2$ J. $|\boldsymbol{F}| = 5$ N, $|\boldsymbol{d}| = \sqrt{5}$ m. $\cos\theta = W/(|\boldsymbol{F}||\boldsymbol{d}|) = 2/(5\sqrt{5})$; $\theta = \arccos\left(2/\sqrt{125}\right) \approx 79.7°$.

Solution 6.8

(a) Constant speed: $P = Mgv = 1500(9.8)(40/60) = 9800\,\text{W} \approx 9.8\,\text{kW}$. (b) After 5 s: $v = 1.0(5) = 5$ m/s. Force needed: $F = Mg + Ma = 1500(10.8) = 16\,200\,\text{N}$. $P_{\max} = Fv = 16200(5) = 81\,\text{kW}$.

Solution 6.9

$W = \int_0^{3d} F_0 e^{-x/d}\mathrm{d}x = F_0 d[-e^{-x/d}]_0^{3d} = F_0 d(1 - e^{-3}) = 0.950\,F_0 d$. Compare: $F_0 \cdot 3d = 3F_0 d$. Ratio: 0.317 (about 32%).

Solution 6.10

(a) Bucket rises h: $W_b = Mgh$. Rope CM rises $h/2$: $W_r = mgh/2$. Total: $W = (M + m/2)gh$.
(b) At height y, mass being lifted is $M + m(1 - y/h)$, speed is $v = P/[(M + m(1 - y/h))g]$. Since $\mathrm{d}y/\mathrm{d}t = v$: $[M+m-my/h]g\,\mathrm{d}y = P\,\mathrm{d}t$. Integrating: $t = g[(M+m)h-mh/2]/P = (M+m/2)gh/P$.

Chapter 7: Kinetic Energy and the Work-Energy Theorem

Solution 7.1

(a) $K = \frac{1}{2}(5.0)(4.0)^2 = 40$ J. (b) $W = 0 - 40 = -40$ J. (c) Double speed to 8.0 m/s: $W = \frac{1}{2}(5.0)(8.0)^2 - 40 = 160 - 40 = 120$ J. (d) From 4.0 to 6.0 m/s: $W = \frac{1}{2}(5.0)(36) - 40 = 90 - 40 = 50$ J. Doubling the speed (part c, $\Delta v = 4$ m/s) costs 120 J, while going from 4.0 to 6.0 m/s (part d, $\Delta v = 2$ m/s) costs 50 J. Note that the remaining increment from 6.0 to 8.0 m/s (also a Δv of 2 m/s) costs $120 - 50 = 70$ J. Because $K \propto v^2$, each additional increment of speed costs more energy than the last.

Solution 7.2

(a) Runner: $K_r = \frac{1}{2}(60)(10)^2 = 3000$ J. Car: $K_c = \frac{1}{2}(1200)(10)^2 = 60{,}000$ J. The car has 20× the kinetic energy.

(b) Runner: $\Delta K_r = \frac{1}{2}(60)(144-100) = 1320$ J. Car: $\Delta K_c = \frac{1}{2}(1200)(144-100) = 26{,}400$ J. The car's ΔK is 20× larger, even though both have the same $\Delta v = 2$ m/s. Since $\Delta K = \frac{1}{2}m(v_f^2 - v_i^2) = \frac{1}{2}m(v_f + v_i)\Delta v$, the factor m appears linearly: 20 times the mass means 20 times the change in kinetic energy for the same Δv.

Solution 7.3

WET: $W_{\text{net}} = mgd\sin 30° = 2.0(9.8)(3.0)(0.5) = 29.4$ J. Then $\frac{1}{2}mv^2 = 29.4$, giving $v = \sqrt{29.4} = 5.42$ m/s.

Kinematic check: $a = g\sin 30° = 4.9$ m/s^2. $v^2 = v_0^2 + 2a\Delta x = 0 + 2(4.9)(3.0) = 29.4$; $v = 5.42$ m/s. ✓

Solution 7.4

(a) $W = \int_0^2 (3x^2 + 2)\mathrm{d}x = [x^3 + 2x]_0^2 = 12$ J. $v = \sqrt{2W/m} = \sqrt{24} = 4.90$ m/s.

(b) Need $\frac{1}{2}(1.0)(3.0)^2 = 4.5$ J of work. $\int_0^x (3x'^2 + 2)\mathrm{d}x' = x^3 + 2x = 4.5$. By inspection or numerical solution: $x^3 + 2x - 4.5 = 0$. Testing $x = 1$: $1 + 2 = 3 < 4.5$. Testing $x = 1.2$: $1.728 + 2.4 = 4.128 < 4.5$. Testing $x = 1.25$: $1.953 + 2.5 = 4.453 \approx 4.5$. So $x \approx 1.25$ m.

Solution 7.5

The ball accelerates from rest to $v = 40$ m/s over $d = 1.5$ m. By the WET: $\bar{F}\,d = \frac{1}{2}mv^2$, so $\bar{F} = mv^2/(2d) = 0.145(1600)/(2 \times 1.5) = 232/3.0 \approx 77$ N.

Solution 7.6

(a) Just before impact: all PE converted to KE. $K_1 = mgh_1 = 0.25(9.8)(2.0) = 4.9$ J.

(b) Just after bounce: KE converts to PE up to $h_2 = 1.5$ m. $K_2 = mgh_2 = 0.25(9.8)(1.5) = 3.675$ J.

(c) Energy lost: $\Delta E = K_1 - K_2 = 4.9 - 3.675 = 1.225$ J.

(d) Fraction lost: $\Delta E/K_1 = 1.225/4.9 = 0.25 = 25\%$.

Solution 7.7

(a) $\frac{1}{2}kx_0^2 = \frac{1}{2}mv^2$; $v = x_0\sqrt{k/m} = 0.10\sqrt{500/0.50} = 0.10\sqrt{1000} = 3.16$ m/s.
(b) $\frac{1}{2}kx_0^2 = mgh$; $h = kx_0^2/(2mg) = 500(0.01)/[2(0.50)(9.8)] = 5/9.8 = 0.510$ m.
(c) Along a 45° ramp, the block rises to the same height $h = 0.510$ m (only gravity and the normal force act; the normal does no work). The distance along the ramp is $d = h/\sin 45° = 0.510/0.707 = 0.721$ m.

Solution 7.8

(a) System approach: $W_{\text{net}} = m_2gd - \mu_k m_1 gd = [5(9.8) - 0.25(3)(9.8)](2.0) = (49 - 7.35)(2.0) = 83.3$ J. $v = \sqrt{2(83.3)/(3+5)} = \sqrt{20.8} = 4.56$ m/s.
(b) Apply the WET to m_2 alone: $W = m_2gd - Td = \frac{1}{2}m_2v^2$. So $T = m_2g - m_2v^2/(2d) = 5(9.8) - 5(20.8)/(2 \times 2.0) = 49 - 26.0 = 23.0$ N.

Solution 7.9

WET: $W = \Delta K$. $W = Fd = mad$. $\Delta K = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$. Setting equal: $mad = \frac{1}{2}m(v^2 - v_0^2)$, so $2ad = v^2 - v_0^2$, i.e., $v^2 = v_0^2 + 2a\Delta x$. ✓

Solution 7.10

The force is $F = F_0$ for $0 \le x < 4d$, $F = 0$ for $4d \le x < 6d$, and $F = -F_0/2$ for $6d \le x \le 8d$.
At $x = 4d$: $W = F_0(4d)$. $\frac{1}{2}(2.0)v^2 = 4F_0d$; $v = \sqrt{4F_0d} = 2\sqrt{F_0d}$.
At $x = 6d$: No additional work from $4d$ to $6d$ ($F = 0$), so $v = 2\sqrt{F_0d}$ (unchanged).
At $x = 8d$: $W_{\text{total}} = 4F_0d + 0 + (-F_0/2)(2d) = 4F_0d - F_0d = 3F_0d$. $\frac{1}{2}(2.0)v^2 = 3F_0d$; $v = \sqrt{3F_0d}$.

Solution 7.11

(a) Bottom of loop: $mgh = \frac{1}{2}mv^2$; $v = \sqrt{2(9.8)(20)} = \sqrt{392} = 19.8$ m/s.
(b) Top of loop (height $2R = 12$ m): $mg(h - 2R) = \frac{1}{2}mv_{\text{top}}^2$; $v_{\text{top}} = \sqrt{2(9.8)(20-12)} = \sqrt{156.8} = 12.5$ m/s.
(c) At the top: $mg + N = mv_{\text{top}}^2/R$; $N = m(v_{\text{top}}^2/R - g) = 50(156.8/6 - 9.8) = 50(26.1 - 9.8) = 50(16.3) = 817$ N.
(d) Minimum: $v_{\text{top,min}}^2 = gR$ (when $N = 0$); $gR = 2g(h_{\min} - 2R)$; $h_{\min} = 2R + R/2 = 5R/2 = 15.0$ m.

Solution 7.12

(a) $\frac{1}{2}mv_0^2 = \mu_k mgd$; $d = v_0^2/(2\mu_k g)$.
(b) Doubling v_0: $d \propto v_0^2$, so d increases by a factor of 4.
(c) Stopping time: $v_0 = at = \mu_k g\,t$, so $t = v_0/(\mu_k g)$. Average power: $P_{\text{avg}} = \Delta K/t = \frac{1}{2}mv_0^2/[v_0/(\mu_k g)] = \frac{1}{2}\mu_k mgv_0$.

Solution 7.13

(a) $F\,\mathrm{d}x = mv\,\mathrm{d}v$: $-bv\,\mathrm{d}x = mv\,\mathrm{d}v$, so $\mathrm{d}x = -(m/b)\mathrm{d}v$. $d = \int_{v_0}^{0} -(m/b)\mathrm{d}v = mv_0/b$.
(b) From $F = m\,\mathrm{d}v/\mathrm{d}t$: $-bv = m\,\mathrm{d}v/\mathrm{d}t$; $\mathrm{d}v/v = -(b/m)\mathrm{d}t$; $\ln(v/v_0) = -bt/m$; $v(t) = v_0e^{-bt/m}$.
(c) $d = \int_0^\infty v_0e^{-bt/m}\mathrm{d}t = v_0(m/b) = mv_0/b$. ✓

Solution 7.14

(a) $W = \int_0^L F_0e^{-x/\lambda}\mathrm{d}x = F_0\big[-\lambda e^{-x/\lambda}\big]_0^L = F_0\lambda(1 - e^{-L/\lambda})$.

(b) $\frac{1}{2}mv^2 = F_0\lambda(1-e^{-L/\lambda})$; $v = \sqrt{2F_0\lambda(1-e^{-L/\lambda})/m}$.

(c) As $L \to \infty$: $W_\infty = F_0\lambda$. $v_\infty = \sqrt{2F_0\lambda/m}$. The force drops off exponentially, so it can only deliver a finite total impulse; the particle reaches a finite limiting speed.

(d) Need $0.9 \times \frac{1}{2}mv_\infty^2 = \frac{1}{2}mv^2$, i.e., $1-e^{-L/\lambda} = 0.9$; $e^{-L/\lambda} = 0.1$; $L = \lambda \ln 10 \approx 2.30\lambda$.

Solution 7.15

(a) $W = \int_0^L F_0 \sin(\pi x/L)\mathrm{d}x = F_0[-\frac{L}{\pi}\cos(\pi x/L)]_0^L = \frac{2F_0 L}{\pi}$.

(b) $v = \sqrt{2W/m} = \sqrt{4F_0 L/(\pi m)}$.

(c) Since $F(x) = F_0 \sin(\pi x/L) > 0$ for all $x \in (0, L)$, the kinetic energy $K(x) = \int_0^x F\,\mathrm{d}x'$ is strictly increasing on $[0, L]$. The speed is therefore maximized at the right endpoint $x = L$. (The hint warns that the usual interior-critical-point condition $F(x) = 0$ with F changing sign does not apply here, because F is positive throughout the interval.)

Solution 7.16

$W = \int_C \boldsymbol{F}_{\text{net}} \cdot \mathrm{d}\boldsymbol{r} = \int_{t_i}^{t_f} m\boldsymbol{a} \cdot \boldsymbol{v}\,\mathrm{d}t$ (using $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$). Key identity: $\boldsymbol{a} \cdot \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{v} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v} \cdot \boldsymbol{v}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(v^2)$. So $W = \int_{t_i}^{t_f} \frac{m}{2}\frac{\mathrm{d}(v^2)}{\mathrm{d}t}\mathrm{d}t = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$. ✓

Solution 7.17

(a) $F\,\mathrm{d}x = v\,\mathrm{d}p$ (since $F = \mathrm{d}p/\mathrm{d}t$ and $\mathrm{d}x = v\,\mathrm{d}t$). With $p = \gamma m v$: $\mathrm{d}p = m\,\mathrm{d}(\gamma v)$. Now use the identity $v\,\mathrm{d}(\gamma v) = c^2\,\mathrm{d}\gamma$, which follows from differentiating $\gamma^2(c^2 - v^2) = c^2$ to get $2\gamma\,\mathrm{d}\gamma\,(c^2 - v^2) - 2\gamma^2 v\,\mathrm{d}v = 0$, hence $v\,\mathrm{d}(\gamma v) = v\gamma\,\mathrm{d}v + v^2\,\mathrm{d}\gamma = c^2\,\mathrm{d}\gamma$. Therefore $F\,\mathrm{d}x = mc^2\,\mathrm{d}\gamma = c^2\,\mathrm{d}(\gamma m)$.

(b) $W = \int mc^2\,\mathrm{d}\gamma = (\gamma_f - \gamma_i)mc^2$. Starting from rest ($\gamma_i = 1$): $W = (\gamma - 1)mc^2 = K_{\text{rel}}$.

(c) For $v \ll c$: $\gamma \approx 1 + \frac{1}{2}v^2/c^2$, so $K \approx \frac{1}{2}mv^2$. ✓

Chapter 8: Potential Energy and Energy Conservation

Solution 8.1

$K_i = \frac{1}{2}mv^2 = \frac{1}{2}(2.0)(15)^2 = 225\,\text{J}$. At max height, $K_f = 0$. Conservation: $\frac{1}{2}mv^2 = mgh$, so $h = v^2/(2g) = 225/(2 \times 10) = 11.25\,\text{m}$.

Solution 8.2

$U_s = \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}(800)(0.05)^2 = 1.0\,\text{J}$. Setting $U_s = \frac{1}{2}mv^2$: $v = \sqrt{2U_s/m} = \sqrt{2(1.0)/0.20} = \sqrt{10} = 3.16\,\text{m/s}$.

Solution 8.3

(a) $mgh_1 = \frac{1}{2}mv^2 + mgh_2$, so $v = \sqrt{2g(h_1 - h_2)}$.

(b) The car clears the hill if $v^2 \geq 0$, i.e., $h_2 \leq h_1$. So $h_{2,\max} = h_1$ (with $v = 0$ at the top).

Solution 8.4

(a) $K_i = mgh_i = 0.25(9.8)(2.0) = 4.9\,\text{J}$. $K_f = mgh_f = 0.25(9.8)(1.5) = 3.675\,\text{J}$. Energy lost: $\Delta E = 4.9 - 3.675 = 1.225\,\text{J}$.

(b) Just before impact: $K_{\text{before}} = 4.9\,\text{J}$. Just after: $K_{\text{after}} = 3.675\,\text{J}$. Fraction lost: $1 - 3.675/4.9 = 1 - 0.75 = 25\%$.

(c) The lost energy goes into thermal energy (heating the ball and floor), sound, and deformation.

Solution 8.5

Spring gives launch speed: $\frac{1}{2}k(\Delta x)^2 = \frac{1}{2}mv_0^2$, so $v_0 = \Delta x\sqrt{k/m}$. The ball is launched horizontally from height H. Fall time: $H = \frac{1}{2}gt^2$, so $t = \sqrt{2H/g}$. Horizontal range: $R = v_0 t = \Delta x\sqrt{k/m} \cdot \sqrt{2H/g} = \Delta x\sqrt{2kH/(mg)}$.

Solution 8.6

(a) At the top of the loop (height $2R$): $mgh = \frac{1}{2}mv_{\text{top}}^2 + mg(2R)$. Minimum when $N = 0$ at top: $mg = mv_{\text{top,min}}^2/R$, so $v_{\text{top,min}}^2 = gR$. Substituting: $gh_{\text{min}} = \frac{1}{2}gR + 2gR$, giving $h_{\text{min}} = \frac{5}{2}R$.

(b) For $h = 3R$: $v_{\text{top}}^2 = 2g(3R - 2R) = 2gR$. At top: $mg + N = mv_{\text{top}}^2/R = 2mg$, so $N = mg$.

(c) At bottom (height 0): $mgh = \frac{1}{2}mv_{\text{bot}}^2$, so $v_{\text{bot}}^2 = 2g(3R) = 6gR$. Radial equation at bottom: $N - mg = mv_{\text{bot}}^2/R = 6mg$, so $N = 7mg$.

Solution 8.7

(a) Height above bottom: $h = \ell(1 - \cos\theta_0)$. Energy conservation: $mg\ell(1 - \cos\theta_0) = \frac{1}{2}mv^2$, so $v = \sqrt{2g\ell(1 - \cos\theta_0)}$.

(b) At bottom, radial equation: $T - mg = mv^2/\ell = 2mg(1 - \cos\theta_0)$. So $T = mg(3 - 2\cos\theta_0)$.

(c) At angle θ: height $= \ell(1 - \cos\theta)$, so $v^2 = 2g\ell(\cos\theta - \cos\theta_0)$. Radial: $T - mg\cos\theta = mv^2/\ell$. So $T = mg(3\cos\theta - 2\cos\theta_0)$.

(d) Set $T_{\text{bot}} = 3mg$: $mg(3 - 2\cos\theta_0) = 3mg$, so $\cos\theta_0 = 0$, giving $\theta_0 = 90°$.

Solution 8.8

(a) For a string, the critical condition is at the top of the circle: $mg = mv_{\text{top}}^2/\ell$ (tension $= 0$), so $v_{\text{top,min}}^2 = g\ell$. Energy conservation from bottom to top (height 2ℓ): $\frac{1}{2}mv_{\text{bot}}^2 = \frac{1}{2}mv_{\text{top}}^2 + mg(2\ell)$, so $v_{\text{bot,min}}^2 = g\ell + 4g\ell = 5g\ell$. Thus $v_{\text{bot,min}} = \sqrt{5g\ell}$.

(b) At the top, $T = 0$ (the string is just barely taut).

(c) With a rigid rod, the rod can push (compressive normal force), so N can be negative. The critical condition becomes $v_{\text{top,min}} = 0$ (just barely reaches the top). Then $v_{\text{bot,min}} = \sqrt{4g\ell} = 2\sqrt{g\ell}$, which is less than the string case.

Solution 8.9

(a) At bottom of ramp: $v = \sqrt{2gh}$. On rough surface: $W_f = -\mu_k mgd = 0 - \frac{1}{2}mv^2 = -mgh$. So $d = h/\mu_k$.

(b) Now $K_i = \frac{1}{2}mv_0^2$ at the top. At bottom of ramp: $K = \frac{1}{2}mv_0^2 + mgh$. On rough surface: $\mu_k mgd = \frac{1}{2}mv_0^2 + mgh$, so $d = \frac{v_0^2}{2\mu_k g} + \frac{h}{\mu_k}$.

Solution 8.10

(a) Ramp length: $d = h/\sin\theta = 30/\sin 25° = 71.0\,\text{m}$. Friction work on ramp: $W_{f,1} = -\mu_k mg\cos\theta \cdot d = -0.10(50)(9.8)\cos 25°(71.0) = -3153\,\text{J}$.

Generalized WET: $W_{f,1} = \frac{1}{2}mv^2 - mgh$, so $\frac{1}{2}(50)v^2 = 50(9.8)(30) - 3153 = 14700 - 3153 = 11\,547\,\text{J}$. $v = \sqrt{2(11547)/50} = 21.5\,\text{m/s}$.

(b) On flat: $\mu_k mg \cdot d_2 = \frac{1}{2}mv^2$, so $d_2 = v^2/(2\mu_k g) = (21.5)^2/(2 \times 0.15 \times 9.8) = 157.4\,\text{m}$.

(c) Initial PE $= mgh = 14\,700\,\text{J}$. Final KE $= 0$. Total thermal energy $= |W_{f,1}| + |W_{f,2}| = 3153 + \frac{1}{2}(50)(21.5)^2 = 3153 + 11547 = 14\,700\,\text{J}$. So 100% of the PE was converted to thermal energy (which must be the case since the skier starts and ends at rest at height 0).

Solution 8.11

$F_x = 2xy + z^2$, $F_y = x^2$, $F_z = 2xz$.

$(\nabla \times \boldsymbol{F})_x = \partial_y F_z - \partial_z F_y = 0 - 0 = 0$. $(\nabla \times \boldsymbol{F})_y = \partial_z F_x - \partial_x F_z = 2z - 2z = 0$. $(\nabla \times \boldsymbol{F})_z = \partial_x F_y - \partial_y F_x = 2x - 2x = 0$.

So $\nabla \times \boldsymbol{F} = \boldsymbol{0}$; the force is conservative.

To find U: from $F_x = -\partial U/\partial x = 2xy + z^2$, integrate: $U = -(x^2 y + xz^2) + f(y, z)$. From $F_y = -\partial U/\partial y = x^2$: $-(-x^2 + \partial f/\partial y) = x^2$, so $\partial f/\partial y = 0$, meaning $f = g(z)$. From $F_z = -\partial U/\partial z = 2xz$: $-(-2xz + g'(z)) = 2xz$, so $g'(z) = 0$, giving $g = C$.

Therefore $U(x, y, z) = -(x^2 y + xz^2) + C$.

Solution 8.12

(a) $U' = U_0(2x/a^2 - 6x^2/a^3) = 0$ gives $x = 0$ and $x = a/3$. $U'' = U_0(2/a^2 - 12x/a^3)$. At $x = 0$: $U'' = 2U_0/a^2 > 0$ (stable). At $x = a/3$: $U'' = -2U_0/a^2 < 0$ (unstable).

(b) $F(x) = -U'(x) = -U_0(2x/a^2 - 6x^2/a^3)$.

(c) At $x = a$: $U(a) = U_0(1 - 2) = -U_0$. $E = -U_0$. For $x > a$, the $-2(x/a)^3$ term makes $U \to -\infty$, so $K = E - U > 0$ for all $x > a$. The particle accelerates rightward and escapes to $+\infty$.

(d) $U(0) = 0$, $U(a/3) = U_0(1/9 - 2/27) = U_0/27$. Need $K_0 \geq U(a/3) - U(0) = U_0/27$.

(e) $k_{\text{eff}} = U''(0) = 2U_0/a^2$. $\omega = \sqrt{2U_0/(ma^2)}$.

Solution 8.13

(a) $U_{\text{hang}} = -mgy^2/(2L)$. At $y_0 = L/4$: $U_i = -mgL/32$. At $y = L$: $U_f = -mgL/2$. Energy conservation: $0 - mgL/32 = \frac{1}{2}mv^2 - mgL/2$. $\frac{1}{2}v^2 = gL/2 - gL/32 = 15gL/32$. $v = \sqrt{15gL/16} = \frac{1}{4}\sqrt{15gL}$.

(b) Friction work: $W_f = -(\mu_k mg/L)\int_{L/4}^{L}(L - y)\,\mathrm{d}y$. $\int_{L/4}^{L}(L - y)\,\mathrm{d}y = [Ly - y^2/2]_{L/4}^{L} = (L^2 - L^2/2) - (L^2/4 - L^2/32) = L^2/2 - 7L^2/32 = 9L^2/32$. So $W_f = -9\mu_k mgL/32$.

Energy: $\frac{1}{2}mv^2 = 15mgL/32 - 9\mu_k mgL/32$, giving $v = \sqrt{gL(15 - 9\mu_k)/16}$.

(c) The rope won't slide if gravity on the hanging portion is insufficient to overcome static friction on the table portion. With $y_0 = L/4$ hanging, weight of hanging part $= mg/4$, friction force $= \mu_k(3mg/4)$. Condition: $mg/4 \leq 3\mu_k mg/4$, so $\mu_k \geq 1/3$.

Solution 8.14

(a) $U'(x) = U_0 \sinh(x/a)/a$. Setting $U' = 0$: $\sinh(x/a) = 0$, so $x = 0$. $U''(0) = U_0 \cosh(0)/a^2 = U_0/a^2 > 0$, confirming stable equilibrium.

(b) $k_{\text{eff}} = U''(0) = U_0/a^2$. $\omega = \sqrt{U_0/(ma^2)}$.

(c) $U(x) = U_0 \cosh(x/a) \geq U_0$ for all x, with minimum U_0 at $x = 0$. As $x \to \pm\infty$, $U \to \infty$. Therefore, for *any* finite E, the particle is confined between two turning points where $U_0 \cosh(x_t/a) = E$. The motion is always bound: the particle can never escape.

Solution 8.15

(a) $U(x) = mgbx^4$ (the gravitational PE at height $y = bx^4$).

(b) $U'(x) = 4mgbx^3$. At $x = 0$: $U' = 0$ (equilibrium). For $x > 0$: $U' > 0$ ($F < 0$, restoring). For $x < 0$: $U' < 0$ ($F > 0$, restoring). Since $U(0)$ is a minimum, it is stable.

(c) $U''(0) = 12mgbx^2\big|_{x=0} = 0$. The effective spring constant vanishes! The leading-order potential near $x = 0$ is $U \approx mgbx^4$, not $\frac{1}{2}kx^2$. Therefore small oscillations are *not* simple harmonic. The restoring force is $F = -4mgbx^3$, which is cubic, not linear. The standard argument fails because the Taylor expansion of U starts at fourth order, not second. (The oscillation period depends on amplitude, unlike SHM.)

Solution 8.16

(a) $E = \frac{1}{2}mv^2 - GMm/r$. At launch: $E = \frac{1}{2}mv_0^2 - GMm/R$. At escape ($r \to \infty$, $v = 0$): $E = 0$. Setting $E = 0$: $\frac{1}{2}mv_e^2 = GMm/R$, giving $v_e = \sqrt{2GM/R}$.

(b) For $v_0 \ll v_e$, max height $r_{\max}$ satisfies $\frac{1}{2}mv_0^2 - GMm/R = -GMm/r_{\max}$. Let $r_{\max} = R + h$ with $h \ll R$: $\frac{1}{r_{\max}} \approx \frac{1}{R}(1 - h/R)$. Substituting: $\frac{1}{2}v_0^2 = GM/R - GM(1-h/R)/R = GMh/R^2 = gh$. So $h = v_0^2/(2g)$.

(c) From $\frac{1}{2}v_0^2 = GM(1/R - 1/r_{\max})$: $1/r_{\max} = 1/R - v_0^2/(2GM) = 1/R - v_0^2/(v_e^2 R)$. $r_{\max} = \frac{R}{1 - v_0^2/v_e^2}$. Height: $h = r_{\max} - R = \frac{Rv_0^2/v_e^2}{1-v_0^2/v_e^2} = \frac{v_0^2 R}{v_e^2 - v_0^2}$.

Solution 8.17

(a) $F_x = y^2z^3$, $F_y = 2xyz^3$, $F_z = 3xy^2z^2$.

$(\nabla \times \boldsymbol{F})_x = \partial_y(3xy^2z^2) - \partial_z(2xyz^3) = 6xyz^2 - 6xyz^2 = 0$. $(\nabla \times \boldsymbol{F})_y = \partial_z(y^2z^3) - \partial_x(3xy^2z^2) = 3y^2z^2 - 3y^2z^2 = 0$. $(\nabla \times \boldsymbol{F})_z = \partial_x(2xyz^3) - \partial_y(y^2z^3) = 2yz^3 - 2yz^3 = 0$.

So $\nabla \times \boldsymbol{F} = \mathbf{0}$. ✓

(b) From $F_x = -\partial U/\partial x = y^2z^3$: $U = -xy^2z^3 + f(y,z)$.

Check F_y: $-\partial U/\partial y = 2xyz^3 - \partial f/\partial y$. Need this $= 2xyz^3$, so $\partial f/\partial y = 0$, meaning $f = g(z)$.

Check F_z: $-\partial U/\partial z = 3xy^2z^2 - g'(z)$. Need this $= 3xy^2z^2$, so $g'(z) = 0$, giving $g = C$.

Therefore $U = -xy^2z^3 + C$.

(c) $-\nabla U = -(-y^2z^3)\,\hat{\boldsymbol{\imath}} - (-2xyz^3)\,\hat{\boldsymbol{\jmath}} - (-3xy^2z^2)\,\hat{\boldsymbol{k}} = y^2z^3\,\hat{\boldsymbol{\imath}} + 2xyz^3\,\hat{\boldsymbol{\jmath}} + 3xy^2z^2\,\hat{\boldsymbol{k}} = \boldsymbol{F}$. ✓

Chapter 9: Linear Momentum and Impulse

Solution 9.1

Taking the initial direction as positive: $J = m(v_f - v_i) = 0.145(-50 - 40) = -13.05$ N s (toward the pitcher). $F_{\text{avg}} = J/\Delta t = 13.05/0.001 = 13\,050$ N.

Solution 9.2

(a) $0 = m_t v_t + m_a v_a$, so $v_a = -5.0(6.0)/70 = -0.43$ m/s (opposite to throw).

(b) $K = \frac{1}{2}m_t v_t^2 + \frac{1}{2}m_a v_a^2 = \frac{1}{2}(5)(36) + \frac{1}{2}(70)(0.184) = 90 + 6.4 = 96.4$ J. This energy came from the astronaut's internal (chemical/muscular) energy: it was not present as KE initially.

Solution 9.3

(a) $J = m(v_f - v_i) = 0.40(-9 - 12) = -8.4$ N s.

(b) $F_{\text{avg}} = -8.4/0.020 = -420$ N (420 N away from wall).

(c) If the ball sticks: $J_{\text{stick}} = m(0 - 12) = -4.8$ N s. The bounce impulse (8.4 N s) is $8.4/4.8 = 1.75$ times larger than the sticking impulse. Bouncing always produces a larger impulse (and force) than sticking.

Solution 9.4

(a) $v = \sqrt{2gh} = \sqrt{2(9.8)(2)} = 6.26$ m/s.

(b) $J = m\Delta v = 60(6.26) = 376$ N s. $F = 376/0.01 = 37\,600$ N $\approx 64\,mg$.

(c) $F = 376/0.5 = 752$ N $\approx 1.3\,mg$. Bending knees reduces the force by a factor of 50.

Solution 9.5

(a) Horizontal momentum conserved: $mV\cos\theta = (m+M)V_f$, so $V_f = mV\cos\theta/(m+M)$.
(b) $K_i = \frac{1}{2}mV^2$, $K_f = \frac{m^2V^2\cos^2\theta}{2(m+M)}$. $\Delta E = \frac{1}{2}mV^2 \cdot \frac{M+m\sin^2\theta}{m+M}$.

Solution 9.6

(a) $J = F_0\tau$. (b) $v_\infty = F_0\tau/m$. (c) $v(t) = \frac{F_0\tau}{m}(1-e^{-t/\tau})$.
(d) $W = \int_0^\infty Fv\,\mathrm{d}t = \frac{F_0^2\tau}{m}\int_0^\infty(e^{-t/\tau}-e^{-2t/\tau})\,\mathrm{d}t = \frac{F_0^2\tau}{m}(\tau-\tau/2) = \frac{F_0^2\tau^2}{2m}$. Check: $\frac{1}{2}mv_\infty^2 = \frac{F_0^2\tau^2}{2m}$. ✓

Solution 9.7

$m_3 = 10-3-4 = 3\,\text{kg}$. x: $3v_{3x} = -4(6)$, $v_{3x} = -8$ m/s. y: $3v_{3y} = -3(8)$, $v_{3y} = -8$ m/s. $v_3 = 8\sqrt{2} \approx 11.3\,\text{m/s}$ at 45° south of west.

Solution 9.8

(a) $V = mv_0/(m+M) = 0.010(400)/2.01 = 1.99\,\text{m/s}$.
(b) $h = V^2/(2g) = (1.99)^2/19.6 = 0.202\,\text{m} \approx 20\,\text{cm}$.
(c) $K_i = \frac{1}{2}(0.010)(400)^2 = 800\,\text{J}$. $K_f = \frac{1}{2}(2.01)(1.99)^2 = 3.98\,\text{J}$. Fraction lost: $(800-3.98)/800 = 99.5\%$.

Solution 9.9

(a) No bounce: $F = \dot{m}v = 2(20) = 40\,\text{N}$.
(b) Elastic: $F = 2\dot{m}v = 80\,\text{N}$.
(c) Deflected at 90°: the component toward the wall changes from v to 0, so $F = \dot{m}v = 40\,\text{N}$ (same as no bounce).

Solution 9.10

(a) $p = Mv_0 = (M+\dot{m}t)v(t)$, so $v(t) = Mv_0/(M+\dot{m}t)$.
(b) $K(t) = \frac{M^2v_0^2}{2(M+\dot{m}t)}$, which decreases since the denominator grows.
(c) $\dot{K} = -\frac{M^2v_0^2\dot{m}}{2(M+\dot{m}t)^2} = -\frac{1}{2}\dot{m}v^2$. Energy is dissipated at rate $\frac{1}{2}\dot{m}v^2$ as heat from the inelastic sand–car collision.

Solution 9.11

(a) From $Mv = (M-|\mathrm{d}M|)(v+\mathrm{d}v) + |\mathrm{d}M|(v-v_e)$: $M\,\mathrm{d}v = v_e\,|\mathrm{d}M|$. With $|\mathrm{d}M| = -\mathrm{d}M$: $\mathrm{d}v = -v_e\,\mathrm{d}M/M$. Integrating: $\Delta v = v_e\ln(M_0/M_f)$.
(b) $M_f = 0.2M_0$: $\Delta v = v_e\ln 5 \approx 1.61v_e$.
(c) $M\dot{v} = v_e\dot{m} - Mg$. Liftoff requires $\dot{v} > 0$, i.e., $v_e\dot{m} > M_0 g$.

Solution 9.12

(a) Free fall: $v = \sqrt{2gy}$.
(b) Weight of pile: $(m/L)yg$. Impact force: $(m/L)v^2 = 2mgy/L$. Total: $F = 3mgy/L$.
(c) At $y = L$: $F = 3mg$. Immediately after the last link lands, the force drops to mg (static weight only).

Solution 9.13

(a) Each stage: $\Delta v_i = -v_e \ln(1-f)$. Two stages: $\Delta v = -2v_e \ln(1-f)$.

(b) A single stage with the same total fuel has $(1-f_{\text{tot}}) = (1-f)^2$, giving the same Δv only if casing mass is zero. With nonzero casing, discarding empty tanks reduces the mass that subsequent stages must accelerate, improving the effective mass ratio.

Solution 9.14

(a) $\mathrm{d}m = \rho_c \pi r^2 v\,\mathrm{d}t$ and $\mathrm{d}m = 4\pi\rho_w r^2 \dot{r}\,\mathrm{d}t$. Equating: $\dot{r} = \rho_c v/(4\rho_w)$.

(b) Try $v = \alpha t$, $r = \beta t^2$. Then $m \propto r^3 \propto t^6$, $mv \propto t^7$. $\frac{\mathrm{d}}{\mathrm{d}t}(mv) = mg$ gives $7(\text{const})t^6 = g(\text{const})t^6$, so $7\alpha = g$ and $a = g/7$.

(c) $\frac{\mathrm{d}}{\mathrm{d}t}(mv) = mg$. Expanding: $m\dot{v} + \dot{m}v = mg$. With $\dot{v} = g/7$: $m(g/7) + \dot{m}v = mg$, so $\dot{m}v = 6mg/7$. Thus 6/7 of the weight goes to accelerating accreted mass; only 1/7 accelerates the drop itself.

Solution 9.15

(a) Full disk: CM at origin, mass M. Hole: radius $R/3$, centered at $(R/3, 0)$, mass $m_h = M(R/3)^2/R^2 = M/9$. Remaining piece: mass $8M/9$. By the negative-mass method: $M(0) = (8M/9)x_{\text{cm}} + (M/9)(R/3)$, so $x_{\text{cm}} = -R/24$. The CM shifts away from the hole.

(b) The impulse-momentum theorem gives $J = (8M/9)\,v_{\text{cm}}$, so $v_{\text{cm}} = 9J/(8M)$ in the $\hat{\boldsymbol{\imath}}$ direction.

Chapter 10: Collisions

Solution 10.1

From $v_2' = \frac{2m_1}{m_1+m_2}V = V/2$: $4m_1 = m_1 + m_2 \Rightarrow m_2 = 3m_1$. Relative velocity reversal: $v_1' = v_2' - V = -V/2$. Verify: $v_1' = \frac{m_1 - 3m_1}{4m_1}V = -V/2$. ✓

Solution 10.2

(a) $mv_0 = (m+M)v_f \Rightarrow v_f = mv_0/(m+M)$.

(b) $h = v_f^2/(2g) = m^2 v_0^2/[2g(m+M)^2]$.

(c) $v_0 = \frac{m+M}{m}\sqrt{2gh}$.

(d) $|\Delta K|/K_i = 1 - m/(m+M) = M/(m+M)$.

Solution 10.3

(a) By the equal-mass perpendicular-velocity theorem: ball 1 deflects at 60° (since 30° + 60° = 90°).

(b) From y-equation: $v_2' = \sqrt{3}\,v_1'$. From x-equation: $v_0 = \frac{1}{2}v_1' + \frac{3}{2}v_1' = 2v_1'$. So $v_1' = v_0/2$ and $v_2' = \sqrt{3}\,v_0/2$. Check energy: $v_1'^2 + v_2'^2 = v_0^2/4 + 3v_0^2/4 = v_0^2$. ✓

Solution 10.4

(a) Round-trip time: $\Delta t = 2L/c$. Impulse per reflection: $\Delta p = 2h\nu/c$. $F = \Delta p/\Delta t = (2h\nu/c)/(2L/c) = h\nu/L$.

(b) At separation L': $F(L') = h\nu/L'$. $W = \int_L^{2L} \frac{h\nu}{L'}\,\mathrm{d}L' = h\nu \ln 2$.

Remark. This result treats the photon frequency as constant during the expansion. A more careful analysis shows that the photon loses energy as it does work on the receding mirrors: each reflection off a moving mirror produces a small Doppler redshift, causing ν to decrease as L increases. The exact relationship (an *adiabatic invariant* $h\nu \cdot L = \text{const}$) is derived in courses on analytical mechanics or statistical mechanics. Accounting for this gives the corrected result $W = h\nu/2$.

Solution 10.5

(a) $v_2' = \frac{2}{1+r}v_0$. $f = \frac{m_2 v_2'^2}{m_1 v_0^2} = \frac{4r}{(1+r)^2}$.

(b) $\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{4(1-r)}{(1+r)^3} = 0$ gives $r = 1$, $f_{\max} = 1$.

(c) $r = 12$: $f = 48/169 \approx 0.284$. After n head-on collisions: $(1-f)^n = 1/e$. $n = -1/\ln(1-f) = -1/\ln(121/169) \approx 3.0$ collisions.

Solution 10.6

Ball hits block at $v_b = \sqrt{2gh}$. Momentum: $mv_b = mv_1' + Mv_2'$. Restitution: $v_2' - v_1' = ev_b$. Solving: $v_1' = \frac{m-eM}{m+M}\sqrt{2gh}$; $v_2' = \frac{m(1+e)}{m+M}\sqrt{2gh}$. (b) $h' = v_1'^2/(2g) = h\left(\frac{m-eM}{m+M}\right)^2$. (c) $m = M$, $e = 0.8$: $v_1' = 0.1\sqrt{2gh}$, $h' = 0.01h$.

Solution 10.7

(a) In lab frame, ball horizontal velocity $= v_0\cos\theta - V$ where V is cannon recoil. Momentum: $0 = m(v_0\cos\theta - V) - MV$; $V = mv_0\cos\theta/(m+M)$. (b) Ball's lab velocity: $v_x = Mv_0\cos\theta/(m+M)$; $v_y = v_0\sin\theta$. Range $= 2v_xv_y/g = Mv_0^2\sin 2\theta/[g(m+M)]$. (c) Since $R \propto \sin 2\theta$, the maximum is at $\theta = 45^\circ$, giving $R_{\max} = Mv_0^2/[g(m+M)]$. For a fixed cannon ($M \to \infty$): $R_{\text{fixed}} = v_0^2/g$. Ratio: $R_{\max}/R_{\text{fixed}} = M/(m+M) < 1$—the recoil reduces the range by the factor $M/(m+M)$.

Solution 10.8

(a) $v_f = \frac{2(5)+3(0)}{5} = 2.0\,\text{m/s}$.

(b) $K_i = \frac{1}{2}(2)(25) = 25\,\text{J}$. $K_f = \frac{1}{2}(5)(4) = 10\,\text{J}$. $\Delta K = -15\,\text{J}$.

(c) $\mu = (2)(3)/(2+3) = 1.2\,\text{kg}$. $\frac{1}{2}\mu v_{\text{rel}}^2 = \frac{1}{2}(1.2)(25) = 15\,\text{J}$. ✓

Solution 10.9

(a) Both arrive at floor with $v_0 = \sqrt{2gh}$. Large ball bounces up at v_0. In the large ball's frame, small ball approaches at $2v_0$; after elastic collision with $M \gg m$, it reverses: leaves at $2v_0$ upward in that frame. Lab frame: $v' = 2v_0 + v_0 = 3v_0$. Height: $h' = (3v_0)^2/(2g) = 9h$.

(b) $M = 2m$: after the bounce, $v_1 = +v_0$ (big ball, up) and $v_2 = -v_0$ (small ball, down). Elastic collision formulas: $v_2' = \frac{2(2m)}{3m}(v_0) + \frac{m-2m}{3m}(-v_0) = \frac{4v_0}{3} + \frac{v_0}{3} = \frac{5v_0}{3}$. Height: $h' = (5v_0/3)^2/(2g) = 25h/9 \approx 2.78h$.

Solution 10.10

(a) At max height, $\boldsymbol{v}_{\text{cm}} = 0$. Momentum: $\mathbf{0} = m\cdot\mathbf{0} + m(2v_0\hat{\boldsymbol{\imath}}) + m\boldsymbol{v}_3$. $\boldsymbol{v}_3 = -2v_0\hat{\boldsymbol{\imath}}$.

(b) Max height: $H = v_0^2/(2g)$. Fall time: $t = v_0/g$. Piece 1 (zero velocity): lands directly below at $x = 0$. Piece 2: $x = 2v_0t = 2v_0^2/g$. Piece 3: $x = -2v_0^2/g$.

(c) $x_{\text{cm}} = \frac{1}{3}(0 + 2v_0^2/g - 2v_0^2/g) = 0$. Unexploded firecracker also lands at $x = 0$ (launched vertically). ✓

Solution 10.11

$K = \sum\frac{1}{2}m_i|\boldsymbol{v}_{\text{cm}} + \boldsymbol{v}_i'|^2 = \frac{1}{2}Mv_{\text{cm}}^2 + \boldsymbol{v}_{\text{cm}}\cdot\sum m_i\boldsymbol{v}_i' + \sum\frac{1}{2}m_i|\boldsymbol{v}_i'|^2$. The cross term: $\sum m_i\boldsymbol{v}_i' = \sum m_i(\boldsymbol{v}_i - \boldsymbol{v}_{\text{cm}}) = M\boldsymbol{v}_{\text{cm}} - M\boldsymbol{v}_{\text{cm}} = \mathbf{0}$. So $K = \frac{1}{2}Mv_{\text{cm}}^2 + K_{\text{int}}$.

For two particles: $v_1^{\text{cm}} = \frac{m_2}{m_1+m_2}v_{\text{rel}}$, $v_2^{\text{cm}} = \frac{-m_1}{m_1+m_2}v_{\text{rel}}$. $K_{\text{int}} = \frac{1}{2}m_1\frac{m_2^2}{(m_1+m_2)^2}v_{\text{rel}}^2 + \frac{1}{2}m_2\frac{m_1^2}{(m_1+m_2)^2}v_{\text{rel}}^2 = \frac{1}{2}\frac{m_1m_2}{m_1+m_2}v_{\text{rel}}^2 = \frac{1}{2}\mu v_{\text{rel}}^2$.

Solution 10.12

(a) If 2 balls exit at $v_0/2$: momentum $= 2m(v_0/2) = mv_0$ ✓, but KE $= 2 \cdot \frac{1}{2}m(v_0/2)^2 = mv_0^2/4 \neq mv_0^2/2$. Only 1 ball at v_0 conserves both momentum and energy.

(b) Two in at v_0: $p = 2mv_0$, $K = mv_0^2$. Two out at v_0: $p = 2mv_0$, $K = mv_0^2$. Both conserved.

(c) In an equal-mass elastic collision, the striker stops and the target takes all its velocity. Each intermediate ball briefly becomes a striker, transfers all momentum to the next, and stops. The impulse propagates as a compression wave at the speed of sound in the ball material.

Chapter 11: Rotation of Rigid Bodies

Solution 11.1

(a) $\omega = 120 \times 2\pi/60 = 4\pi \approx 12.57\,\mathrm{rad/s}$.
(b) $v = R\omega = 0.30(12.57) = 3.77\,\mathrm{m/s}$.
(c) $a_c = R\omega^2 = 0.30(12.57)^2 = 47.4\,\mathrm{m/s^2}$.

Solution 11.2

$\omega_f = 1200 \times 2\pi/60 = 40\pi \approx 125.7\,\mathrm{rad/s}$.
(a) $\alpha = 125.7/6.0 = 20.9\,\mathrm{rad/s^2}$.
(b) $\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2}(20.9)(36) = 377\,\mathrm{rad} = 60.0$ revolutions.
(c) $a_{\tan} = R\alpha = 0.15(20.9) = 3.14\,\mathrm{m/s^2}$. $a_c = R\omega^2 = 0.15(125.7)^2 = 2370\,\mathrm{m/s^2}$.

Solution 11.3

$I_{\mathrm{cm}} = \frac{2}{5}MR^2$. Tangent axis is distance $d = R$ from center. $I = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$.

Solution 11.4

(a) $M = \frac{2\pi\sigma_0}{R^2}\int_0^R r^3\,\mathrm{d}r = \frac{\pi\sigma_0 R^2}{2}$.
(b) $I = \frac{2\pi\sigma_0}{R^2}\int_0^R r^5\,\mathrm{d}r = \frac{\pi\sigma_0 R^4}{3}$. With $\sigma_0 = 2M/(\pi R^2)$: $I = \frac{2}{3}MR^2$.
(c) Uniform disk: $I = \frac{1}{2}MR^2$. Since $2/3 > 1/2$, the non-uniform disk has larger I—the $\sigma \propto r^2$ density concentrates mass at larger radii where the r^2 weighting amplifies its contribution.

Solution 11.5

(a) Model the plate as a stack of thin rods parallel to the x-axis: $I_x = \int y^2\,\mathrm{d}m$. By symmetry and direct integration: $I_x = Ma^2/12$. Similarly $I_y = Ma^2/12$.

(b) $I_z = I_x + I_y = Ma^2/6$.

(c) Edge axis is parallel to the center axis and displaced by $d = a/2$: $I_{\mathrm{edge}} = Ma^2/12 + M(a/2)^2 = Ma^2/12 + Ma^2/4 = Ma^2/3$.

Solution 11.6

$\tau = FR = 10(0.20) = 2.0\,\mathrm{N\,m}$. $\alpha = \tau/I = 2.0/0.50 = 4.0\,\mathrm{rad/s^2}$.

(a) $\alpha = 4.0\,\mathrm{rad/s^2}$. (b) $\omega = 4.0(4.0) = 16\,\mathrm{rad/s}$. (c) $K = \frac{1}{2}(0.50)(256) = 64\,\mathrm{J}$. (d) $\theta = \frac{1}{2}(4.0)(16) = 32\,\mathrm{rad}$. $W = \tau\theta = 2.0(32) = 64\,\mathrm{J}$. ✓

Solution 11.7

(a) $K_{\mathrm{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2}(MR^2)(20)^2 = \frac{1}{2}(1.5)(0.1225)(400) = 36.75\,\mathrm{J}$.

(b) $v = R\omega = 0.35(20) = 7.0\,\text{m/s}$. $K_{\text{trans}} = \frac{1}{2}Mv^2 = \frac{1}{2}(1.5)(49) = 36.75\,\text{J}$. $K_{\text{total}} = 36.75 + 36.75 = 73.5\,\text{J}$.

(c) Fraction rotational: $36.75/73.5 = 1/2 = 50\%$. (For a hoop, rotational and translational KE are always equal when rolling.)

Solution 11.8

Slice into disks at height z from center. Disk radius: $r = \sqrt{R^2 - z^2}$, thickness $\mathrm{d}z$, mass $\mathrm{d}m = \rho\pi r^2 \mathrm{d}z$. Each disk: $I_{\text{disk}} = \frac{1}{2}r^2\,\mathrm{d}m$. Integrate:

$I = \frac{\rho\pi}{2}\int_{-R}^{R}(R^2 - z^2)^2\,\mathrm{d}z = \frac{\rho\pi}{2}\int_{-R}^{R}(R^4 - 2R^2z^2 + z^4)\,\mathrm{d}z = \frac{\rho\pi}{2}[2R^5 - \frac{4R^5}{3} + \frac{2R^5}{5}] = \frac{8\pi\rho R^5}{15}$.

With $M = \frac{4}{3}\pi\rho R^3$: $I = \frac{2}{5}MR^2$.

Solution 11.9

(a) CM at $L/2$, distance from pivot: $d = |x - L/2|$. $I(x) = \frac{1}{12}ML^2 + M(x - L/2)^2$.

(b) $\mathrm{d}I/\mathrm{d}x = 2M(x - L/2) = 0$ gives $x = L/2$. $I_{\min} = \frac{1}{12}ML^2$.

(c) $I(0) = \frac{1}{12}ML^2 + M(L/2)^2 = \frac{1}{3}ML^2$. This is 4 times larger than $I_{\min}$, because all the mass on the "long side" of the pivot is at maximum distance.

Solution 11.10

At height z above the apex, the cone's radius is $r(z) = Rz/h$. A disk at height z: $\mathrm{d}m = \rho\pi r^2\,\mathrm{d}z = \rho\pi R^2 z^2/h^2\,\mathrm{d}z$. Total mass: $M = \rho\pi R^2 h/3$.

$I = \int_0^h \frac{1}{2}r^2\,\mathrm{d}m = \frac{\rho\pi R^4}{2h^4}\int_0^h z^4\,\mathrm{d}z = \frac{\rho\pi R^4}{2h^4}\cdot\frac{h^5}{5} = \frac{\rho\pi R^4 h}{10}$.

With $\rho = 3M/(\pi R^2 h)$: $I = \frac{3}{10}MR^2$.

Chapter 12: Dynamics of Rotational Motion

Solution 12.1

(a) $\omega_1 = 3.0(8.0) = 24\,\text{rad/s}$.

(b) $\theta_1 = \frac{1}{2}(3.0)(64) = 96$ rad. $\theta_2 = 24(5) = 120$ rad. $\theta_3 = 24(10)/2 = 120$ rad. Total: $336\,\text{rad}$.

(c) $\alpha_3 = -24/10 = -2.4\,\text{rad/s}^2$.

Solution 12.2

(a) $M = \frac{2\pi\sigma_0}{R^2}\int_0^R r^3\,\mathrm{d}r = \frac{\pi\sigma_0 R^2}{2}$.

(b) $I = \frac{2\pi\sigma_0}{R^2}\int_0^R r^5\,\mathrm{d}r = \frac{\pi\sigma_0 R^4}{3}$. With $\sigma_0 = 2M/(\pi R^2)$: $I = \frac{2}{3}MR^2$.

(c) Uniform: $I = \frac{1}{2}MR^2$. The non-uniform disk has larger I because $\sigma \propto r^2$ concentrates mass at larger radii.

Solution 12.3

(a) If $T_1 = T_2$, there is no net torque on the pulley and no angular acceleration.

(b) $T_1 - m_1 g = m_1 a$; $m_2 g - T_2 = m_2 a$; $(T_2 - T_1) = \frac{1}{2}Ma$. Adding: $a = (m_2 - m_1)g/(m_1 + m_2 + M/2)$.

(c) $T_1 = m_1(g + a)$, $T_2 = m_2(g - a)$.

(d) $(m_2 - m_1)gh = \frac{1}{2}(m_1 + m_2 + M/2)v^2$; $v = \sqrt{2(m_2 - m_1)gh/(m_1 + m_2 + M/2)}$.

(e) $a = 30/9 = 3.33\,\text{m/s}^2$. $T_1 = 26.7\,\text{N}$. $T_2 = 33.3\,\text{N}$. $v = \sqrt{10} = 3.16\,\text{m/s}$.

Solution 12.4

(a) $a = \frac{2}{3} g \sin\theta$.
(b) $f_s = \frac{1}{3} Mg \sin\theta \le \mu_s Mg \cos\theta$: $\mu_s \ge \frac{1}{3}\tan\theta$.
(c) $MgL\sin\theta = \frac{3}{4}Mv^2$; $v = \sqrt{4gL\sin\theta/3}$.
(d) No friction $\Rightarrow$ no torque; ω stays constant. The cylinder slides while spinning.
(e) Only translational KE converts to PE: $h = v^2/(2g)$. Translational KE is $\frac{2}{3}$ of total: $h = \frac{2}{3}L\sin\theta$.
(f) $h = \frac{2}{3}(2)\sin 30^\circ = 2/3 \approx 0.667\,\mathrm{m}$.

Solution 12.5

(a) $I = \frac{1}{3}ML^2$; $\tau = MgL/2$; $\alpha_0 = 3g/(2L)$.
(b) $\omega = \sqrt{3g\sin\theta/L}$.
(c) $L_{\text{ang}} = I\omega = \frac{M}{3}\sqrt{3gL^3\sin\theta}$.
(d) $\mathrm{d}L/\mathrm{d}\theta \propto \cos\theta/\sqrt{\sin\theta} = 0$ at $\theta = 90^\circ$.
(e) $\alpha_0 = 3(9.8)/(2\times 1.2) = 12.25\,\mathrm{rad/s^2}$. $\omega(90^\circ) = \sqrt{3(9.8)/1.2} = 4.95\,\mathrm{rad/s}$. $L_{\max} = \frac{0.5}{3}\sqrt{3(9.8)(1.2)^3} = 1.19\,\mathrm{kg\,m^2/s}$.

Solution 12.6

(a) $a = 5g\sin\theta/7$.
(b) $f_s = \frac{2}{7}Mg\sin\theta \le \mu_s Mg\cos\theta$: $\mu_s \ge \frac{2}{7}\tan\theta$.
(c) $v_{\text{sphere}} = \sqrt{10gh/7}$. Sliding block: $\sqrt{2gh}$. Hollow cylinder ($c = 1$): $\sqrt{gh}$. Ranking: block > sphere > cylinder.

Solution 12.7

(a) $Mg - T = Ma$, $TR = \frac{1}{2}MR^2(a/R)$, so $T = \frac{1}{2}Ma$ and $a = 2g/3$.
(b) $T = Mg/3$.
(c) $a/g = 2/3 \approx 67\%$ of free-fall acceleration. For a thin axle of radius $r \ll R$: $a = g/(1 + R^2/(2r^2)) \approx 2gr^2/R^2 \to 0$ as $r \to 0$—a very thin axle makes the yo-yo descend extremely slowly because nearly all the energy goes into rotation.

Solution 12.8

(a) $v = v_0 - \mu_k g t$, $\omega = 5\mu_k g t/(2R)$. Rolling at $v = R\omega$: $t_r = 2v_0/(7\mu_k g)$.
(b) $v_f = v_0 - \mu_k g t_r = 5v_0/7$.
(c) $K_f = \frac{7}{10}Mv_f^2 = \frac{7}{10}M(25v_0^2/49) = \frac{5Mv_0^2}{14}$. Fraction lost: $1 - (5/14)/(1/2) = 1 - 5/7 = 2/7 \approx 29\%$.

Solution 12.9

$I = \frac{1}{2}(5.0)(0.0625) = 0.156\,\mathrm{kg\,m^2}$. $\omega_f = 33(2\pi/60) = 3.46\,\mathrm{rad/s}$. $\alpha = 3.46/4 = 0.864\,\mathrm{rad/s^2}$.
(a) $\tau = I\alpha = 0.135\,\mathrm{N\,m}$.
(b) $\theta = \frac{1}{2}\alpha t^2 = 6.91$ rad. $W = \tau\theta = 0.933\,\mathrm{J}$.
(c) $K_f = \frac{1}{2}I\omega_f^2 = 0.934\,\mathrm{J}$. ✓

Chapter 13: Angular Momentum

Solution 13.1

(a) $L = (I_{\text{disk}} + I_{\text{person}})\omega_0 = (\frac{1}{2}MR^2 + mR^2)\omega_0$.

(b) At center, $I_f = \frac{1}{2}MR^2$. Conservation: $\omega_1 = \frac{M+2m}{M}\omega_0$.

(c) $m = M$: $\omega_1/\omega_0 = 3$.

(d) $K_f/K_i = I_i/I_f = (M+2m)/M = 3$ (for $m = M$). The person does internal work walking inward against the centrifugal tendency, converting chemical energy to kinetic energy.

Solution 13.2

(a) At any instant, $L = rmv\sin\phi$. But $r\sin\phi = d$ (perpendicular distance from O to the line of motion), which is constant. So $L = mvd$.

(b) Since v, m, and d are all constant, $L = mvd$ is constant.

(c) No force acts ($\boldsymbol{F} = \mathbf{0}$), so $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = \mathbf{0}$ and $\mathrm{d}\boldsymbol{L}/\mathrm{d}t = \mathbf{0}$. ✓

Solution 13.3

(a) The pivot exerts a large impulsive force during the collision, so linear momentum is not conserved. But this force acts *at* the pivot ($r_\perp = 0$), producing zero torque about the pivot. Angular momentum about the pivot is conserved.

(b) $mv_0L = (\frac{1}{3}ML^2 + mL^2)\omega$, so $\omega = \frac{3mv_0}{(M+3m)L}$.

(c) After the collision, energy conservation gives $\frac{1}{2}I\omega^2 = (M+m)gd_{\mathrm{cm}}(1-\cos\theta_{\max})$, where $I = \frac{1}{3}ML^2 + mL^2$ and $d_{\mathrm{cm}} = \frac{ML/2+mL}{M+m} = \frac{(M+2m)L}{2(M+m)}$. Solving: $\cos\theta_{\max} = 1 - \frac{I\omega^2}{2(M+m)gd_{\mathrm{cm}}}$.

Solution 13.4

(a) $L = I\omega_0 = 4.0\times10^{-4}(60\pi) = 0.0754\,\mathrm{kg\,m^2/s}$.

(b) $\alpha = \tau_f/I = 0.012/(4.0\times10^{-4}) = 30\,\mathrm{rad/s^2}$.

(c) $t = \omega_0/\alpha = 60\pi/30 = 2\pi \approx 6.28\,\mathrm{s}$.

(d) $\theta = \omega_0 t - \frac{1}{2}\alpha t^2 = 60\pi(2\pi) - \frac{1}{2}(30)(4\pi^2) = 60\pi^2 \approx 592\,\mathrm{rad} \approx 94$ revolutions.

Solution 13.5

(a) Linear momentum conserved: $v_{\mathrm{cm}} = mv_0/(m+M)$.

(b) Combined CM at $x_{\mathrm{cm}} = ML/(2(m+M))$ from the struck end. $L_{\mathrm{cm}} = mv_0 \cdot ML/(2(m+M)) = mMv_0L/(2(m+M))$. $I_{\mathrm{cm}} = \frac{1}{12}ML^2 + M[mL/(2(m+M))]^2 + m[ML/(2(m+M))]^2 = ML^2(4m+M)/(12(m+M))$. $\omega = 6mv_0/((4m+M)L)$.

(c) $K_f = \frac{1}{2}(m+M)v_{\mathrm{cm}}^2 + \frac{1}{2}I_{\mathrm{cm}}\omega^2$. Fraction lost: $1 - K_f/K_i$.

(d) For $m = M$: $v_{\mathrm{cm}} = v_0/2$, $\omega = 6v_0/(5L)$, $I_{\mathrm{cm}} = 5ML^2/24$. $K_f = Mv_0^2/4 + 3Mv_0^2/20 = 2Mv_0^2/5$. Lost: $1 - (2/5)/(1/2) = 20\%$.

Solution 13.6

(a) $\omega = 2mv/((M+2m)R) = 2(30)(4)/((100+60)(2)) = 240/320 = 0.75\,\mathrm{rad/s}$.

(b) $K_i = \frac{1}{2}(30)(16) = 240\,\mathrm{J}$. $K_f = \frac{1}{2}(\frac{1}{2}(100)(4)+30(4))(0.75)^2 = \frac{1}{2}(320)(0.5625) = 90\,\mathrm{J}$. Fraction lost: $150/240 = 62.5\%$.

(c) The lost energy was converted to heat and sound during the inelastic "collision" of the child landing on the disk.

Solution 13.7

(a) $T_f = T(R_f/R)^2 = 30\text{ days} \times (10^4/(7\times10^8))^2 = 2.592\times10^6 \times 2.04\times10^{-10} = 5.3\times10^{-4}\,\mathrm{s} \approx 0.53\,\mathrm{ms}$.

(b) $K_f/K_i = (R/R_f)^2 = (7\times10^4)^2 = 4.9\times10^9 \approx 5\times10^9$.

(c) Gravitational potential energy released during collapse.

(d) $v_{\rm eq} = 2\pi R_f/T_f = 2\pi(10^4)/(5.3\times10^{-4}) \approx 1.2\times10^8$ m/s $\approx 0.4c$. Mildly relativistic, a full treatment would need special-relativistic corrections.

Solution 13.8

(a) $I\omega_0 = 2I\omega_f$, so $\omega_f = \omega_0/2$.

(b) $K_f/K_i = 1/2$. Fraction lost: 50%.

(c) Heat from kinetic friction between the sliding surfaces as the disks reach a common ω.

(d) Perfectly *inelastic*: angular momentum is conserved but kinetic energy is not—exactly analogous to two equal masses sticking together, where 50% of the KE is also lost.

Solution 13.9

(a) $\boldsymbol{\tau} = \boldsymbol{r}\times F(r)\hat{\boldsymbol{r}} = F(r)(\boldsymbol{r}\times\hat{\boldsymbol{r}}) = \boldsymbol{0}$ since $\boldsymbol{r} = r\hat{\boldsymbol{r}} \parallel \hat{\boldsymbol{r}}$.

(b) $\mathrm{d}\boldsymbol{L}/\mathrm{d}t = \boldsymbol{\tau} = \boldsymbol{0}$, so $\boldsymbol{L} = \text{const}$.

(c) $\boldsymbol{L} = m\boldsymbol{r}\times\boldsymbol{v} = m(r\hat{\boldsymbol{r}})\times(\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = mr^2\dot{\theta}(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{\theta}}) = mr^2\dot{\theta}\,\hat{\boldsymbol{k}}$.

(d) $\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta$, so $\mathrm{d}A/\mathrm{d}t = \frac{1}{2}r^2\dot{\theta} = L/(2m) = \text{const}$.

Solution 13.10

(a) Initial total $L = 0$ (everything at rest). After the throw, the ball (at the rim, moving tangentially) has $L_{\rm ball} = m_b v_0 R$. Conservation: $0 = m_b v_0 R + (\frac{1}{2}MR^2 + mR^2)\omega$, so $\omega = -\frac{m_b v_0}{(\frac{1}{2}M+m)R}$. The turntable+person rotates opposite to the ball's motion.

(b) $K_f = \frac{1}{2}m_b v_0^2 + \frac{1}{2}(\frac{1}{2}MR^2 + mR^2)\omega^2 > 0 = K_i$. KE increased; the energy came from the person's muscles (internal chemical energy).

(c) A radially thrown ball has $\boldsymbol{v}\parallel\boldsymbol{r}$ from the axis, so $L_{\rm ball} = m_b(\boldsymbol{r}\times\boldsymbol{v}) = 0$. By conservation, $L_{\rm turntable} = 0$ and the turntable does not rotate.

Chapter 14: Rigid Body Equilibrium

Solution 14.1

(a) Torques about the hinge: $TL\sin\theta = Mg(L/2) + mgL$, so $T = \frac{(M/2+m)g}{\sin\theta}$.

(b) $H_x = T\cos\theta = (M/2+m)g\cot\theta$. $V + T\sin\theta = (M+m)g \Rightarrow V = Mg/2$.

(c) T minimized when $\sin\theta$ is maximized: $\theta = 90^\circ$, $T_{\rm min} = (M/2+m)g$.

(d) $T = 60(10)/\sin 30^\circ = 1200\,\mathrm{N}$. $H_x = 60(10)\cot 30^\circ = 600\sqrt{3} \approx 1039\,\mathrm{N}$. $V = 100\,\mathrm{N}$.

Solution 14.2

(a) Torques about left support: $N_R L = Mg(L/2) + mgd$. $N_R = g(M/2 + md/L)$. $N_L = (M+m)g - N_R = g(M/2 + m(1-d/L))$.

(b) $N_L + N_R = g(M/2 + m - md/L + M/2 + md/L) = (M+m)g$. ✓

(c) Equal load: $N_L = N_R = (M+m)g/2$. From $N_R = g(M/2+md/L)$: $(M+m)/2 = M/2 + md/L$, so $d = mL/(2m) = L/2$. The person stands at the center (same as the plank's CM).

Solution 14.3

(a) $N_w = \frac{Mg}{2}\cot\theta$. $f_s = N_w = \frac{Mg}{2}\cot\theta$. $N_f = Mg$.

(b) $f_s \le \mu_s N_f$: $\frac{Mg}{2}\cot\theta \le \mu_s Mg$; $\theta_{\rm min} = \arctan(1/(2\mu_s))$.

(c) With the person at fraction f of the way up: the person's weight mg acts at distance fL from the base. Torques about base: $N_w L\sin\theta = Mg(L/2)\cos\theta + mg(fL)\cos\theta$. $N_w = (M/2+mf)g\cot\theta$.

Slip condition: $(M/2 + mf)\cot\theta \le \mu_s(M+m)$, so $\theta_{\min} = \arctan\left(\frac{M/2+mf}{(M+m)\mu_s}\right)$. The minimum angle increases as the person climbs higher (f increases), so the ladder is more likely to slip when the person is near the top.

Solution 14.4

Cable is horizontal, attached at $L/2$, arm at angle θ. The cable's moment arm about the pivot is $(L/2)\sin\theta$. Torques about pivot: $T(L/2)\sin\theta = Mg(L/2)\cos\theta + mgL\cos\theta$. $T = (M+2m)g\cot\theta$.

Pivot: $P_x = T = (M+2m)g\cot\theta$ (horizontal, toward cable). $P_y = (M+m)g$ (vertical, upward). $P = \sqrt{P_x^2 + P_y^2}$ at angle $\arctan(P_y/P_x)$ from horizontal.

Solution 14.5

(a) $\theta_{\text{slide}} = \arctan\mu_s$.

(b) The cube tips when the CM's vertical projection passes over the downhill edge. For a cube: $\theta_{\text{tip}} = 45°$.

(c) Tips first if $\theta_{\text{tip}} < \theta_{\text{slide}}$: $45° < \arctan\mu_s$, i.e., $\mu_s > 1$. For $\mu_s < 1$: slides first.

Solution 14.6

(a) Half-cable angle: $\tan\alpha = 2h/d$. Vertical equilibrium at midpoint: $2T\sin\alpha = mg$. $T = mg/(2\sin\alpha)$. For small sag ($h \ll d$): $\sin\alpha \approx 2h/d$, so $T \approx mgd/(4h)$.

(b) $T = 20(9.8)(12)/(4\times 0.5) = 2352/2 = 1176\,\text{N}$, about 6 times the weight.

(c) As $h \to 0$: $\sin\alpha \to 0$ and $T \to \infty$. A perfectly horizontal cable cannot have a vertical force component (since $T\sin 0 = 0$), so it cannot support any weight. Real cables must always sag.

Solution 14.7

(a) Torques about toes: $2F_h(L-d) = Mg(L/2)$. $F_h = MgL/(4(L-d))$.

(b) For $d \ll L$: $F_h \approx Mg/4$; each hand supports about 25% of the body weight. The feet carry the remaining 50%.

(c) Moving hands farther from shoulders (increasing d) decreases $L-d$, increasing F_h. The moment arm of the hands about the toes decreases, so each hand must push harder to balance the same gravitational torque.

Solution 14.8

(a) Forces: Mg (down, at center), N ($\perp$ to incline, at contact), f_s (along incline, at contact), T (horizontal, at top).

(b) Torques about contact point (eliminates N and f_s): $T(R + R\cos\theta) = MgR\sin\theta$. $T = \frac{Mg\sin\theta}{1+\cos\theta}$.

(c) $\sum F_x = 0$: $T\cos\theta + f_s\cos\theta = N\sin\theta$ (along incline direction). More directly: resolve along and perpendicular to incline. $\perp$: $N = Mg\cos\theta + T\sin\theta$. $\|$: $f_s = Mg\sin\theta - T\cos\theta$.

Solution 14.9

(a) Torques about pivot: $m_A g d_A = m_B g d_B$, so $m_A d_A = m_B d_B$.

(b) $F_{\text{pivot}} = (M + m_A + m_B)g$ (supports the total weight; the plank's weight acts at the pivot, contributing zero torque but adding to the vertical load).

(c) $d_B = m_A d_A/m_B = 25(2.0)/40 = 1.25\,\text{m}$.

Solution 14.10

(a) Torques about pin: $T \sin\alpha \cdot L = Mg(L/3) + mgx$. $T(x) = \frac{Mg/3+mgx/L}{\sin\alpha} = \frac{(M/3)g+(mx/L)g}{\sin\alpha}$.

(b) $P_x = T\cos\alpha$. $P_y = (M+m)g - T\sin\alpha$.

(c) T is minimized at $x = 0$ (load at the pin): $T_{\min} = Mg/(3\sin\alpha)$. T is maximized at $x = L$: $T_{\max} = (M/3+m)g/\sin\alpha$.

(d) $P_x = T\cos\alpha = 0$ only if $T = 0$, which requires $M/3 + mx/L = 0$—impossible for positive masses. So P_x never vanishes; there is always a horizontal pin force (equal to the horizontal component of the cable tension).

Solution 14.11

(a) $A = \pi(d/2)^2 = \pi(0.001)^2 = 3.14\times10^{-6}\,\mathrm{m}^2$. $\sigma = mg/A = 25(9.8)/(3.14\times10^{-6}) = 7.8\times10^7\,\mathrm{Pa} = 78\,\mathrm{MPa}$.

(b) $\varepsilon = \sigma/E = 78\times10^6/(200\times10^9) = 3.9\times10^{-4}$. $\Delta L = \varepsilon L_0 = 3.9\times10^{-4}\times3.0 = 1.2\,\mathrm{mm}$.

(c) $F_{\max} = \sigma_{\mathrm{UTS}}\cdot A = 500\times10^6\times3.14\times10^{-6} = 1570\,\mathrm{N}$. $m_{\max} = 1570/9.8 = 160\,\mathrm{kg}$.

Solution 14.12

(a) The bar is rigid, so both wires stretch by different amounts. For wire i: $\Delta L_i = F_iL_0/(E_iA)$. The bar stays horizontal only if both wires stretch equally: $F_s/(E_sA) = F_a/(E_aA)$, so $F_s/F_a = E_s/E_a = 200/70 = 20/7$. With $F_s + F_a = Mg$: $F_s = \frac{20}{27}Mg \approx 74\%$; $F_a = \frac{7}{27}Mg \approx 26\%$. The stiffer wire carries more load.

(b) For equal stretch with different areas: $F_s/(E_sA_s) = F_a/(E_aA_a)$. For equal load ($F_s = F_a = Mg/2$): $1/(E_sA_s) = 1/(E_aA_a)$, so $A_s/A_a = E_a/E_s = 70/200 = 7/20$. The steel wire can be thinner because it is stiffer.

Chapter 15: Gravitation

Solution 15.1

$F = GM_EM_M/r^2 = (6.674\times10^{-11})(5.97\times10^{24})(7.35\times10^{22})/(3.84\times10^8)^2 = 1.98\times10^{20}\,\mathrm{N}$. This is about 2.0×10^{19} kg-weight ($\approx 2\times10^{16}$ metric tons)—roughly the weight of a small mountain.

Solution 15.2

(a) $g_V = G(0.815M_E)/(0.949R_E)^2 = (0.815/0.9006)g_E = 0.905(9.8) = 8.87\,\mathrm{m/s^2}$.

(b) Mass of rock: $m = 75/9.8 = 7.65\,\mathrm{kg}$. Weight on Venus: $7.65(8.87) = 67.9\,\mathrm{N}$.

Solution 15.3

$g(R+h) = g_s/2$: $(R+h)^2 = 2R^2$, so $h = (\sqrt{2}-1)R = 0.414(6370) = 2640\,\mathrm{km}$.

At ISS altitude ($h = 400$ km): $g = g_s(R/(R+h))^2 = 9.8(6370/6770)^2 = 8.7\,\mathrm{m/s^2}$ (89% of surface g). Astronauts feel weightless because they and the station are in free fall together—there is no contact force (no “floor pushing up”), so the apparent weight is zero.

Solution 15.4

(a) $\frac{1}{2}v_0^2 - g_sR = -g_sR^2/(R+h)$. Solving: $h = v_0^2R/(2g_sR - v_0^2)$.

(b) For $v_0^2 \ll 2g_sR$: $h \approx v_0^2/(2g_s)$. ✓

(c) As $h\to\infty$: $2g_sR - v_0^2 \to 0^+$, so $v_0 \to \sqrt{2g_sR} = v_e$.

(d) $h = (25\times10^6)(6.37\times10^6)/((2)(9.8)(6.37\times10^6) - 25\times10^6) \approx 1595\,\mathrm{km}$. Flat-Earth: $v_0^2/(2g) = 1276\,\mathrm{km}$ (underestimates by $\sim 20\%$).

Solution 15.5

(a) $M(r) = M_p(r/R)^3$. $F = GM_pmr/R^3 = mg_sr/R$ (directed toward center: $F = -mg_sr/R$).
(b) $m\ddot{r} = -mg_sr/R$: SHM with $\omega^2 = g_s/R$, $T = 2\pi\sqrt{R/g_s}$.
(c) Surface orbit: $GM_p/R^2 = v^2/R$, so $v = \sqrt{g_sR}$, $T_{\text{orb}} = 2\pi R/v = 2\pi\sqrt{R/g_s}$. Same. ✓
(d) Energy: $\frac{1}{2}v_c^2 = \frac{1}{2}g_sR$, so $v_c = \sqrt{g_sR}$.
(e) $T = 2\pi\sqrt{6.37 \times 10^6/9.8} = 5066\,\text{s} \approx 84\,\text{min}$. $v_c = \sqrt{(9.8)(6.37 \times 10^6)} = 7.9\,\text{km/s}$.

Solution 15.6

Each neighbor exerts $F = Gm^2/a^2$. By symmetry, net force toward centroid. Component from each: $F\cos 30° = F\sqrt{3}/2$. Total: $F_{\text{net}} = 2F\cos 30° = Gm^2\sqrt{3}/a^2$.

Solution 15.7

(a) $M = 4\pi\rho_0\int_0^R r^2(1 - r/R)\text{d}r = 4\pi\rho_0(R^3/3 - R^3/4) = \pi\rho_0R^3/3$.
(b) $M(r) = 4\pi\rho_0(r^3/3 - r^4/(4R))$. $g(r) = GM(r)/r^2 = 4\pi G\rho_0(r/3 - r^2/(4R))$.
(c) $\text{d}g/\text{d}r = 4\pi G\rho_0(1/3 - r/(2R)) = 0$ at $r = 2R/3$. Physically: at small r, adding enclosed mass dominates and g grows; at large r, the density drops to zero and g falls. The maximum is at $r = 2R/3$.

Solution 15.8

Build the sphere shell by shell. When radius is r: $M(r) = \frac{4}{3}\pi\rho r^3$. Add shell $\text{d}m = 4\pi\rho r^2\text{d}r$.
$\text{d}U = -GM(r)\text{d}m/r = -\frac{16\pi^2G\rho^2}{3}r^4\text{d}r$.
$U = -\frac{16\pi^2G\rho^2}{3}\int_0^R r^4\text{d}r = -\frac{16\pi^2G\rho^2R^5}{15}$.
With $\rho = 3M/(4\pi R^3)$: $U = -3GM^2/(5R)$.

Solution 15.9

$V_{\text{eff}}(r) = -GMm/r + L^2/(2mr^2)$. At circular orbit r_0: $V'_{\text{eff}}(r_0) = 0$ gives $GMm/r_0^2 = L^2/(mr_0^3)$.
$V''_{\text{eff}}(r_0) = -2GMm/r_0^3 + 3L^2/(mr_0^4)$. Using the circular-orbit condition to eliminate L^2: $V''_{\text{eff}} = -2GMm/r_0^3 + 3GMm/r_0^3 = GMm/r_0^3$.
Radial oscillation frequency: $\omega_r^2 = V''_{\text{eff}}/(m) = GM/r_0^3 = \omega_{\text{orb}}^2$. Same frequency: the orbit closes after one period (no precession for $1/r^2$ force).

Solution 15.10

Energy conservation (using reduced mass $\mu = m_1m_2/(m_1 + m_2)$):
$\frac{1}{2}\mu\dot{r}^2 = Gm_1m_2(1/r - 1/d)$.
$\dot{r} = -\sqrt{2G(m_1 + m_2)(1/r - 1/d)}$ (negative because r decreases).
$t = \int_d^0 \frac{\text{d}r}{|\dot{r}|}$. Substitution $r = d\sin^2\eta$: $\text{d}r = 2d\sin\eta\cos\eta\,\text{d}\eta$; $1/r - 1/d = \cos^2\eta/(d\sin^2\eta)$. The integral becomes $t = 2\sqrt{d^3/(2GM)}\int_0^{\pi/2}\sin^2\eta\,\text{d}\eta$, where $M = m_1 + m_2$.
Using $\int_0^{\pi/2}\sin^2\eta\,\text{d}\eta = \pi/4$:

$$t = \frac{\pi}{2}\sqrt{\frac{d^3}{2G(m_1 + m_2)}}.$$

This equals $T_{\text{Kepler}}/2$ for a degenerate elliptical orbit ($e = 1$) with semi-major axis $a = d/2$: $T = 2\pi\sqrt{a^3/(G(m_1 + m_2))} = 2\pi\sqrt{d^3/(8G(m_1 + m_2))}$, so $T/2 = \pi\sqrt{d^3/(8G(m_1 + m_2))} = \frac{\pi}{2}\sqrt{d^3/(2G(m_1 + m_2))}$. ✓

Chapter 16: Kepler's Laws and Orbital Mechanics

Solution 16.1

(a) $r = (GM_E T^2/(4\pi^2))^{1/3} = (3.986 \times 10^{14} \times 86400^2/(4\pi^2))^{1/3} = 4.22 \times 10^7$ m.
(b) $v = 2\pi r/T = 2\pi(4.22 \times 10^7)/86400 = 3.07$ km/s.
(c) $h = r - R_E = 4.22 \times 10^7 - 6.37 \times 10^6 = 3.58 \times 10^7$ m $= 35\,800$ km.

Solution 16.2

$r_p = a(1-e) = 1.496 \times 10^{11}(0.9833) = 1.471 \times 10^{11}$ m. $r_a = a(1+e) = 1.521 \times 10^{11}$ m.
Vis-viva: $v_p = \sqrt{GM_\odot(2/r_p - 1/a)} = 30.3$ km/s; $v_a = \sqrt{GM_\odot(2/r_a - 1/a)} = 29.3$ km/s.

Solution 16.3

(a) $GM_p = v_A^2 r_A = 4800^2(7 \times 10^7) = 1.61 \times 10^{15}$. $v_B = \sqrt{GM_p/r_B} = 7330$ m/s.
(b) $T_A = 2\pi r_A/v_A = 91\,600$ s ≈ 25.4 hr. $T_B = 2\pi r_B/v_B = 25\,700$ s ≈ 7.1 hr.
(c) $T_A^2/T_B^2 = (91600/25700)^2 = 12.7$. $(r_A/r_B)^3 = (7/3)^3 = 12.7$. ✓

Solution 16.4

(a) $r_1 = R_E + 300$ km $= 6.67 \times 10^6$ m, $r_2 = R_E + 35800$ km $= 4.22 \times 10^7$ m. $a_t = (r_1 + r_2)/2 = 2.44 \times 10^7$ m.

(b) $v_1 = \sqrt{GM_E/r_1} = 7.73$ km/s. $v_{t1} = \sqrt{GM_E(2/r_1 - 1/a_t)} = 10.16$ km/s. $\Delta v_1 = 10.16 - 7.73 = 2.43$ km/s. $v_2 = \sqrt{GM_E/r_2} = 3.07$ km/s. $v_{t2} = \sqrt{GM_E(2/r_2 - 1/a_t)} = 1.60$ km/s. $\Delta v_2 = 3.07 - 1.60 = 1.47$ km/s.

(c) $\Delta v_{\text{total}} = 2.43 + 1.47 = 3.90$ km/s. $t = \pi\sqrt{a_t^3/(GM_E)} = 1.90 \times 10^4$ s ≈ 5.3 hr.

Solution 16.5

(a) Each star orbits CM at radius d. Gravitational force: $GM^2/(2d)^2 = Mv^2/d$, so $v = \sqrt{GM/(4d)}$. $T = 2\pi d/v = 4\pi\sqrt{d^3/(GM)}$.

(b) $K = 2 \times \frac{1}{2}Mv^2 = GM^2/(4d)$. $U = -GM^2/(2d)$. $E = -GM^2/(4d)$.

(c) Place test mass at midpoint + radial displacement ϵ. Net force $\propto +\epsilon$ (away from midpoint). Unstable.

(d) Perpendicular displacement ϵ: restoring force $\propto -\epsilon$. SHM with $\omega = \sqrt{2GM/d^3}$.

Solution 16.6

(a) $GM_E = 4\pi^2 r_M^3/T_M^2 = 4\pi^2(3.84 \times 10^8)^3/(2.36 \times 10^6)^2 = 4.01 \times 10^{14}$ m^3/s^2.

(b) $r = R_E + 200$ km $= 6.57 \times 10^6$ m. $T = 2\pi\sqrt{r^3/(GM_E)} = 2\pi\sqrt{(6.57 \times 10^6)^3/4.01 \times 10^{14}} = 5300$ s ≈ 88 min.

(c) $r = (GM_E T^2/(4\pi^2))^{1/3} = (4.01 \times 10^{14}(5400)^2/(4\pi^2))^{1/3} = 6.65 \times 10^6$ m. $h = r - R_E = 280$ km.

Solution 16.7

(a) $r_p = R_E + 500 = 6.87 \times 10^6$ m, $r_a = R_E + 5000 = 1.137 \times 10^7$ m. $a = (r_p + r_a)/2 = 9.12 \times 10^6$ m. $e = (r_a - r_p)/(r_a + r_p) = (1.137 - 0.687)/(1.137 + 0.687) \times 10^7/(1.824 \times 10^7) = 0.247$.

(b) $v_p = \sqrt{GM_E(2/r_p - 1/a)} = 8.56$ km/s. $v_a = \sqrt{GM_E(2/r_a - 1/a)} = 5.17$ km/s.

(c) $T = 2\pi\sqrt{a^3/(GM_E)} = 8670$ s ≈ 144 min.

Solution 16.8

(a) Vis-viva at the burn point ($r = r_0$, $v = v_0 + \Delta v$): $(v_0 + \Delta v)^2 = GM(2/r_0 - 1/a')$. With $v_0^2 = GM/r_0$, solving gives $a' = r_0/(2 - r_0(v_0 + \Delta v)^2/(GM))$.

(b) The burn point is the periapsis of the new orbit, so $r_p = r_0$ and $r_a = 2a' - r_0$.

(c) For $\Delta v \ll v_0$: $(v_0 + \Delta v)^2 \approx v_0^2 + 2v_0\Delta v = GM/r_0 + 2v_0\Delta v$. Then $r_0(v_0 + \Delta v)^2/(GM) \approx 1 + 2\Delta v/v_0$, so $a' = r_0/(1 - 2\Delta v/v_0) \approx r_0(1 + 2\Delta v/v_0)$. Therefore $r_a = 2a' - r_0 = r_0 + 4r_0\Delta v/v_0$, giving $\Delta r_a \approx 4r_0(\Delta v/v_{\text{orb}})$.

Solution 16.9

(a) $\Delta a = 2GM\Delta r/r^3 = 2G(1.4 \times 2 \times 10^{30})(1.8)/r^3 = 6.72 \times 10^{20}/r^3$ m/s^2.

(b) $\Delta a = 9.8$: $r = (6.72 \times 10^{20}/9.8)^{1/3} = (6.86 \times 10^{19})^{1/3} = 4.1 \times 10^6\,\text{m} = 4100\,\text{km}$.

(c) Roche limit: $d_R = 2.44R_{\text{NS}}(\rho_{\text{NS}}/\rho_{\text{human}})^{1/3}$. $\rho_{\text{NS}} \approx M/(\frac{4}{3}\pi R^3) = 2.8 \times 10^{30}/(\frac{4}{3}\pi(10^4)^3) = 6.7 \times 10^{17}\,\text{kg/m}^3$. $d_R = 2.44(10^4)(6.7 \times 10^{17}/10^3)^{1/3} = 2.44 \times 10^4 \times 8.7 \times 10^4 = 2.1 \times 10^9$ m—much larger than the answer in (b), because the Roche limit accounts for the entire body's disruption, not just the acceleration felt.

Solution 16.10

(a) $M_\odot = 4\pi^2a^3/(GT^2) = 4\pi^2(1.496 \times 10^{11})^3/((6.674 \times 10^{-11})(3.156 \times 10^7)^2) = 1.99 \times 10^{30}\,\text{kg}$.

(b) $T^2 \propto a^3$: $(T/T_E)^2 = (a/a_E)^3$. $a = a_E(T/T_E)^{2/3} = 1\ \text{AU} \times 5^{2/3} = 2.92\,\text{AU}$.

(c) $M_J = 4\pi^2a_{\text{Io}}^3/(GT_{\text{Io}}^2) = 4\pi^2(4.22 \times 10^8)^3/((6.674 \times 10^{-11})(1.53 \times 10^5)^2) = 1.90 \times 10^{27}\,\text{kg}$.

Chapter 17: Black Holes

Solution 17.1

$R_s = 2GM/c^2$.

(a) Sun: $R_s = 2(6.674\times10^{-11})(2.0\times10^{30})/(3\times10^8)^2 = 2.67\times10^{20}/(9\times10^{16}) = 2960\,\text{m} \approx 3.0\,\text{km}$. About the size of a small town.

(b) Earth: $R_s = 2(6.674 \times 10^{-11})(6.0 \times 10^{24})/(9 \times 10^{16}) = 8.0 \times 10^{14}/(9 \times 10^{16}) = 8.9\,\text{mm}$. Smaller than a marble.

(c) $10M_\odot$: $R_s = 10 \times 3.0 = 30\,\text{km}$. About the size of a city.

Solution 17.2

(a) $M = 4\times10^6(2\times10^{30}) = 8\times10^{36}\,\text{kg}$. $R_s = 2(6.674\times10^{-11})(8\times10^{36})/(9\times10^{16}) = 1.19\times10^{10}\,\text{m}$. In AU: $1.19 \times 10^{10}/1.5 \times 10^{11} = 0.079\,\text{AU}$. This is about one-fifth of Mercury's orbital radius: the entire black hole fits well inside Mercury's orbit.

(b) $V = \frac{4}{3}\pi(1.19 \times 10^{10})^3 = 7.03 \times 10^{30}\,\text{m}^3$. $\bar{\rho} = 8 \times 10^{36}/7.03 \times 10^{30} = 1.14 \times 10^6\,\text{kg/m}^3$.

(c) This is about 10^3 times the density of water. Much less extreme than stellar-mass black holes ($\bar{\rho} \sim 10^{17}$ kg/m^3), illustrating that $\bar{\rho} \propto 1/M^2$—supermassive black holes are surprisingly "sparse" on average.

Solution 17.3

(a) At $r = R_s = 2GM/c^2$:

$$\Delta a = \frac{2GM\Delta r}{R_s^3} = \frac{2GM\Delta r}{(2GM/c^2)^3} = \frac{2GM\Delta r \cdot c^6}{8G^3M^3} = \frac{c^6\Delta r}{4G^2M^2}.$$

Since $\Delta a \propto 1/M^2$, larger black holes have weaker tidal forces at the horizon.

(b) $M = 10M_\odot = 2 \times 10^{31}$ kg, $\Delta r = 2$ m. $c^6 = 7.29 \times 10^{50}$, $G^2 = 4.454 \times 10^{-21}$, $M^2 = 4 \times 10^{62}$.

$\Delta a = 7.29\times10^{50}\times2/(4\times4.454\times10^{-21}\times4\times10^{62}) = 1.458\times10^{51}/(7.13\times10^{42}) = 2.0\times10^{8}\,\text{m/s}^2 \approx 2\times10^{7}g$.

Instantly lethal—spaghettification occurs well before reaching the horizon.

(c) $\Delta a \propto 1/M^2$: $\Delta a = 2\times10^{8}\times(2\times10^{31}/(8\times10^{36}))^2 = 2\times10^{8}\times6.25\times10^{-12} = 1.25\times10^{-3}\,\text{m/s}^2 \approx 10^{-4}\,g$. Completely negligible: the astronaut would not feel anything unusual while crossing.

(d) $R_s \propto M$, so the horizon of a massive black hole is far from the center. At distance R_s, the gravitational field varies slowly (tidal force $\propto M/R_s^3 \propto 1/M^2$), making the local environment nearly uniform—close to free fall with negligible tidal stress.

Solution 17.4

(a) $\bar{\rho} = M/(\frac{4}{3}\pi R_s^3) = M/(\frac{4}{3}\pi(2GM/c^2)^3) = \frac{3c^6}{32\pi G^3 M^2} \propto 1/M^2$.

(b) $M = 10M_\odot = 2\times10^{31}$ kg: $\bar{\rho} = 3(7.29\times10^{50})/(32\pi(2.97\times10^{-31})(4\times10^{62})) = 2.19\times10^{51}/(1.19\times10^{34}) \approx 1.8\times10^{17}\,\text{kg/m}^3$. This is comparable to nuclear density—matter inside a stellar-mass black hole (if we could see it) would be packed as tightly as the interior of an atomic nucleus.

(c) Set $\bar{\rho} = 1000\,\text{kg/m}^3$: $M^2 = 3c^6/(32\pi G^3\times1000)$. $M = \sqrt{3(7.29\times10^{50})/(32\pi(2.97\times10^{-31})(10^3))} = \sqrt{2.19\times10^{51}/2.97\times10^{-26}} \approx \sqrt{7.4\times10^{76}} \approx 2.7\times10^{38}$ kg $\approx 1.4\times10^{8}M_\odot$. A supermassive black hole of about 10^8 solar masses has the average density of water.

Solution 17.5

(a) $v_e = c/2$: $\frac{1}{2}m(c/2)^2 = GMm/r$, so $mc^2/8 = GMm/r$ and $r = 8GM/c^2 = 4R_s$.

(b) For $10M_\odot$: $4R_s = 4\times30 = 120\,\text{km}$.

(c) At $r = 2R_s$: $v_e = \sqrt{2GM/r} = \sqrt{2GM/(2\cdot2GM/c^2)} = \sqrt{c^2/2} = c/\sqrt{2} \approx 0.71c$.

Solution 17.6

$\Delta\nu/\nu = GM/(Rc^2)$.

(a) Neutron star: $M = 1.4(2\times10^{30}) = 2.8\times10^{30}\,\text{kg}$, $R = 1\times10^{4}\,\text{m}$. $\Delta\nu/\nu = (6.674\times10^{-11})(2.8\times10^{30})/((10^4)(9\times10^{16})) = 1.869\times10^{20}/(9\times10^{20}) = 0.208 \approx 21\%$.

(b) Sun: $\Delta\nu/\nu = (6.674\times10^{-11})(2\times10^{30})/((6.96\times10^{8})(9\times10^{16})) = 1.335\times10^{20}/(6.264\times10^{25}) = 2.13\times10^{-6} \approx 2$ ppm.

(c) Ratio: $0.208/(2.13\times10^{-6}) \approx 10^5$. The neutron star's redshift is $\sim10^5$ times larger because its surface is so much closer to its Schwarzschild radius ($R_{\text{NS}} \approx 3.4\,R_s$ vs. $R_\odot \approx 2.4\times10^{5}\,R_s$).

Solution 17.7

(a) $r_{\text{ISCO}} = 3R_s = 6GM/c^2$. $v = \sqrt{GM/r} = \sqrt{GM\cdot c^2/(6GM)} = c/\sqrt{6} \approx 0.408c \approx 1.22\times10^{8}\,\text{m/s}$.

(b) $M = 2\times10^{31}$ kg. $r = 6(6.674\times10^{-11})(2\times10^{31})/(9\times10^{16}) = 8.9\times10^{4}\,\text{m} = 89\,\text{km}$. $T = 2\pi r/v = 2\pi(8.9\times10^4)/(1.22\times10^8) = 4.6\times10^{-3}\,\text{s} \approx 4.6\,\text{ms}$. Material at the ISCO orbits hundreds of times per second.

(c) $M = 4\times10^{6}M_\odot = 8\times10^{36}$ kg: $r_{\text{ISCO}} = 6(6.674\times10^{-11})(8\times10^{36})/(9\times10^{16}) = 3.6\times10^{10}\,\text{m} \approx 3.6\times10^{7}\,\text{km} \approx 0.24$ AU.

Solution 17.8

$R_s = 2GM/c^2 \Rightarrow M = R_s c^2/(2G)$.

(a) $R_s = 1.5\times10^{11}$ m: $M = (1.5\times10^{11})(9\times10^{16})/(2\times6.674\times10^{-11}) = 1.35\times10^{28}/1.335\times10^{-10} = 1.01\times10^{38}\,\text{kg} \approx 5\times10^{7}M_\odot$. About the mass of a moderately large supermassive black hole.

(b) $R_s = 4.4\times10^{26}$ m: $M = (4.4\times10^{26})(9\times10^{16})/(1.335\times10^{-10}) = 2.97\times10^{53}$ kg $\approx 1.5\times10^{23} M_\odot$.

(c) The observable universe contains $\sim 10^{53}$ kg of ordinary matter, which is comparable to this value. The Schwarzschild radius of the observable universe is roughly the size of the observable universe itself, a coincidence that has intrigued cosmologists (though it does *not* mean we live inside a black hole, because the geometry of an expanding universe is very different from a static Schwarzschild spacetime).

Solution 17.9

(a) $T_H = \hbar c^3/(8\pi GMk_B) = (1.055\times10^{-34})(2.7\times10^{25})/(8\pi(6.674\times10^{-11})(2\times10^{30})(1.381\times10^{-23})) = 2.85\times10^{-9}/(4.63\times10^{-2}) = 6.2\times10^{-8}$ K. This is about 4×10^7 times colder than the cosmic microwave background (2.7 K)—Hawking radiation from stellar-mass black holes is utterly negligible.

(b) $T_H = \hbar c^3/(8\pi Gk_B)\times 1/M \propto 1/M$. Halving the mass doubles the temperature.

(c) $L = \sigma_{\mathrm{SB}}(4\pi R_s^2)T_H^4$. Since $R_s \propto M$ and $T_H \propto 1/M$: $L \propto M^2\times M^{-4} = M^{-2} \propto 1/M^2$. Evaporation time $t_{\mathrm{evap}} \sim Mc^2/L \propto M/M^{-2} = M^3$. The exact result (from GR) is $t_{\mathrm{evap}} \approx 5120\pi G^2M^3/(\hbar c^4) \sim 10^{67}(M/M_\odot)^3$ years—far longer than the age of the universe for any astrophysical black hole.

Solution 17.10

(a) $g_s = GM/R_s^2 = GM/(2GM/c^2)^2 = GMc^4/(4G^2M^2) = c^4/(4GM)$.

(b) For $M_\odot$: $g_s = (3\times10^8)^4/(4(6.674\times10^{-11})(2\times10^{30})) = 8.1\times10^{33}/(5.34\times10^{20}) = 1.5\times10^{13}$ m/s^2. In units of $g_\oplus$: $1.5\times10^{13}/9.8 \approx 1.5\times10^{12}\, g_\oplus$—about a trillion times Earth's gravity.

(c) $g_s \propto 1/M$: more massive black holes have *weaker* surface gravity. This is consistent with Problem 17.3: both g_s and the tidal acceleration decrease with M (though with different power laws: $g_s \propto 1/M$ vs. $\Delta a \propto 1/M^2$), because R_s grows linearly with M, placing the horizon farther from the singularity.

Chapter 18: Harmonic Motion

Solution 18.1

(a) $\omega_0 = \sqrt{32/0.50} = 8$ rad/s. $T = 2\pi/8 = \pi/4 \approx 0.785$ s. $f \approx 1.27$ Hz.

(b) Substitution: $\ddot{x} = -A\omega_0^2\cos(\omega_0 t) = -\omega_0^2 x$. ✓

(c) $v_{\max} = A\omega_0 = 0.15\times8 = 1.2$ m/s at $x = 0$.

(d) $E = \frac{1}{2}kA^2 = \frac{1}{2}(32)(0.0225) = 0.36$ J.

Solution 18.2

(a) Buoyancy = weight: $\rho gAd_0 = Mg \Rightarrow d_0 = M/(\rho A)$.

(b) Displaced by x: $F_{\mathrm{net}} = \rho gAx$ (restoring). $M\ddot{x} = -\rho gAx$: SHM with $\omega = \sqrt{\rho gA/M}$.

(c) $T = 2\pi\sqrt{M/(\rho gA)} = 2\pi\sqrt{d_0/g}$.

(d) $d_0 = 2.0/(1000\times0.008) = 0.25$ m. $\omega = \sqrt{9.8/0.25} = 6.26$ rad/s. $T = 2\pi\sqrt{0.25/9.8} \approx 1.00$ s.

Solution 18.3

(a) $I = \frac{1}{3}ML^2 + mL^2 + \frac{1}{2}mr^2$.

(b) $d = \frac{M(L/2)+mL}{M+m} = \frac{L(M/2+m)}{M+m}$.

(c) $T = 2\pi\sqrt{\frac{I}{(M+m)gd}} = 2\pi\sqrt{\frac{\frac{1}{3}ML^2+mL^2+\frac{1}{2}mr^2}{(M/2+m)gL}}$.

(d) As $m \to 0$: $T \to 2\pi\sqrt{ML^2/3 \cdot 2/(MgL)} = 2\pi\sqrt{2L/(3g)}$. ✓

(e) $I = 0.1067 + 0.192 + 0.000375 = 0.2991$ kg·m^2. $(M + m)gd = 7.84 \times 0.55 = 4.312$ N·m. $T = 2\pi\sqrt{0.2991/4.312} = 2\pi(0.2633) \approx 1.655\,\text{s}$.

Solution 18.4

(a) $\gamma = 0.50/0.80 = 0.625\,\text{s}^{-1}$. $\omega_0 = \sqrt{50} = 7.071\,\text{rad/s}$. $\omega_d = \sqrt{50 - 0.391} = 7.044\,\text{rad/s}$. Underdamped ($\gamma < \omega_0$).

(b) $t_{1/2} = \ln 2/(2\gamma) = 0.693/1.25 = 0.554\,\text{s}$.

(c) $Q = 7.071/1.25 = 5.66$. $A_{\max} \approx (F_0/k) \cdot Q = 0.10 \times 5.66 = 0.566\,\text{m}$.

Solution 18.5

(a) $I = \frac{1}{2}MR^2$. EOM: $I\ddot{\theta} = -\kappa\theta - \beta\dot{\theta}$. So $\ddot{\theta} + \frac{\beta}{I}\dot{\theta} + \frac{\kappa}{I}\theta = 0$. $\omega_0 = \sqrt{2\kappa/(MR^2)}$, $\gamma = \beta/(MR^2)$.

(b) $I = \frac{1}{2}(0.80)(0.01) = 0.004$ kg·m^2. $\omega_0 = \sqrt{125} = 11.18\,\text{rad/s}$. $\gamma = 0.010/0.008 = 1.25\,\text{s}^{-1}$. Underdamped ($\gamma \ll \omega_0$).

(c) $\omega_d = \sqrt{125 - 1.5625} = 11.11\,\text{rad/s}$. $Q = 11.18/2.50 = 4.47$.

(d) Amplitude reaches $1/e$ at $t = 1/\gamma = 0.80\,\text{s}$. Cycles: $n = \omega_d/(2\pi\gamma) = 11.11/7.85 \approx 1.41$.

Solution 18.6

Displaced x: each spring exerts $-kx$. Total: $F = -2kx$. $\omega = \sqrt{2k/m}$; $T = 2\pi\sqrt{m/(2k)}$.

Solution 18.7

(a) Inside the tunnel, the component of gravity along the tunnel is $-g_s x/R$ (same as through the center!). SHM with $T = 2\pi\sqrt{R/g} \approx 84$ min, independent of α. (b) This is the same period as the full-diameter tunnel and the surface-skimming orbit.

Solution 18.8

(a) $kx_0 = mg$; $x_0 = mg/k$. (b) Let $\xi = x - x_0$: $m\ddot{\xi} = -k\xi$. $\omega = \sqrt{k/m}$, identical to horizontal. Gravity shifts the equilibrium but does not change the frequency. (c) $E = \frac{1}{2}k\xi^2 + \frac{1}{2}m\dot{\xi}^2$ (measured from new equilibrium).

Solution 18.9

$\omega = \sqrt{k/m} = 5$ rad/s. (a) $T = 2\pi/5 = 1.26$ s. (b) $v_{\max} = A\omega = 0.50$ m/s. (c) $a_{\max} = A\omega^2 = 2.5$ m/s^2. (d) $v = \omega\sqrt{A^2 - x^2} = 5\sqrt{0.0075} = 0.433$ m/s.

Solution 18.10

Series: $1/k_{\text{eff}} = 1/100 + 1/200 = 3/200$; $k_{\text{eff}} = 66.7$ N/m. $T = 2\pi\sqrt{0.50/66.7} = 0.544$ s. Parallel: $k_{\text{eff}} = 300$ N/m; $T = 2\pi\sqrt{0.50/300} = 0.257$ s.

Solution 18.11

(a)–(b) CM moves on circle of radius $\mathcal{R} - R$. Let θ = angular displacement from bottom. Height: $h \approx \frac{1}{2}(\mathcal{R} - R)\theta^2$. Rolling constraint: $\dot{\phi} = (\mathcal{R} - R)\dot{\theta}/R$. Energy:

$E = \frac{3}{4}M(\mathcal{R} - R)^2\dot{\theta}^2 + \frac{1}{2}Mg(\mathcal{R} - R)\theta^2$.

SHM with $\omega^2 = 2g/[3(\mathcal{R} - R)]$.

(c) A sliding point mass: $\omega^2_{\text{slide}} = g/(\mathcal{R} - R)$. The rolling cylinder oscillates more slowly by factor $\sqrt{2/3}$ because part of the gravitational PE goes into rotational KE, effectively increasing the system's inertia.

Part III

Appendices

Appendix A

Mathematical Foundations

This appendix collects the mathematical tools used throughout this textbook. The material here is for reference; the physics that uses these tools is developed in the main chapters.

A.1 Calculus Review

A.1.1 Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(e^{ax}) = ae^{ax}, \tag{A.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x, \tag{A.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x. \tag{A.3}$$

Chain rule: $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x)) \cdot g'(x)$. Product rule: $(fg)' = f'g + fg'$. Quotient rule: $(f/g)' = (f'g - fg')/g^2$.

A.1.2 Integrals

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \ (n \neq -1), \qquad \int \frac{\mathrm{d}x}{x} = \ln|x| + C, \tag{A.4}$$

$$\int e^{ax} \,\mathrm{d}x = \frac{1}{a}e^{ax} + C, \qquad \int \frac{\mathrm{d}x}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a} + C, \tag{A.5}$$

$$\int \sin x \,\mathrm{d}x = -\cos x + C, \qquad \int \cos x \,\mathrm{d}x = \sin x + C. \tag{A.6}$$

Integration by parts: $\int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u$. Useful when the integrand is a product of two functions, one of which simplifies upon differentiation (e.g., $\int x \sin x \,\mathrm{d}x$ with $u = x$, $\mathrm{d}v = \sin x \,\mathrm{d}x$).

Substitution: If $x = g(t)$, then $\int f(x)\,\mathrm{d}x = \int f(g(t))\, g'(t)\,\mathrm{d}t$. This is the reverse of the chain rule and is used extensively in orbit and collision integrals.

A.1.3 Useful Definite Integrals

$$\int_0^\infty e^{-ax} \,\mathrm{d}x = \frac{1}{a}, \qquad \int_0^\infty x\, e^{-ax} \,\mathrm{d}x = \frac{1}{a^2}, \tag{A.7}$$

$$\int_0^\infty x^n e^{-ax} \,\mathrm{d}x = \frac{n!}{a^{n+1}}, \qquad \int_0^\pi \sin\theta \,\mathrm{d}\theta = 2, \tag{A.8}$$

$$\int_0^\pi \sin^2\theta \,\mathrm{d}\theta = \frac{\pi}{2}, \qquad \int_0^{2\pi} \cos^2\theta \,\mathrm{d}\theta = \pi. \tag{A.9}$$

Gaussian integrals (used in statistical mechanics and some gravitational problems):

$$\int_0^\infty e^{-ax^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \qquad \int_0^\infty x^2 e^{-ax^2}\,\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}. \tag{A.10}$$

A.2 Taylor Series and Approximations

The Taylor expansion of $f(x)$ about $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Important Maclaurin series ($a = 0$):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{A.11}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad (\text{for small } x : \sin x \approx x) \tag{A.12}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad (\text{for small } x : \cos x \approx 1 - x^2/2) \tag{A.13}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \qquad (\text{for small } x : (1+x)^n \approx 1 + nx) \tag{A.14}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad (|x| < 1) \tag{A.15}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad (|x| < 1) \tag{A.16}$$

Applications to physics: The small-angle approximation $\sin\theta \approx \theta$ makes the simple pendulum solvable. The binomial approximation $(1+x)^n \approx 1+nx$ is used to derive the flat-Earth gravitational PE from the universal form: $-GMm/(R+h) \approx \text{const} + mgh$ when $h \ll R$.

A.3 Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}, \tag{A.17}$$

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta, \tag{A.18}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta. \tag{A.19}$$

Sum-to-product (used for beats, Chapter 18):

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right), \tag{A.20}$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right). \tag{A.21}$$

Product-to-sum:

$$\cos A\cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)], \tag{A.22}$$

$$\sin A\sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]. \tag{A.23}$$

A.4 Dimensional Analysis

Every physical equation must be **dimensionally consistent**: both sides must have the same dimensions. The three fundamental mechanical dimensions are length [L], mass [M], and time [T].

Checking equations. If you derive $v = \sqrt{2gh}$, verify: $[v] = \mathrm{L/T}$, $[\sqrt{2gh}] = \sqrt{(\mathrm{L/T^2})(\mathrm{L})} = \sqrt{\mathrm{L^2/T^2}} = \mathrm{L/T}$. ✓

Estimating unknown formulas. If the period T of a simple pendulum depends only on ℓ, m, and g, then $T = C\,\ell^\alpha m^\beta g^\gamma$ for some dimensionless constant C. Matching dimensions: $\mathrm{T} = \mathrm{L}^\alpha\,\mathrm{M}^\beta\,(\mathrm{L\,T^{-2}})^\gamma$. This gives $\beta = 0$ (period is independent of mass!), $\gamma = -1/2$, and $\alpha = 1/2$. Therefore $T = C\sqrt{\ell/g}$—dimensional analysis determines the functional form up to a dimensionless constant (here $C = 2\pi$).

A.5 First-Order Linear ODEs

The equation $\mathrm{d}v/\mathrm{d}t + av = b$ (constant coefficients) has general solution:

$$v(t) = \frac{b}{a} + \left(v_0 - \frac{b}{a}\right) e^{-at}. \tag{A.24}$$

This is used for viscous drag problems: $m\dot{v} = mg - bv$ gives $v(t) = v_T(1 - e^{-t/\tau})$.

For the separable equation $\mathrm{d}v/\mathrm{d}t = f(v)$: separate as $\mathrm{d}v/f(v) = \mathrm{d}t$ and integrate both sides.

A.6 Second-Order Linear ODEs with Constant Coefficients

The general form is $\ddot{x} + b\dot{x} + cx = f(t)$. The homogeneous solution comes from the **characteristic equation** $r^2 + br + c = 0$:

- Two distinct real roots $r_1 \neq r_2$: $x_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
- Repeated root $r_1 = r_2 = r$: $x_h = (C_1 + C_2 t)e^{rt}$.
- Complex conjugate roots $r = \alpha \pm i\beta$: $x_h = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$.

SHM: $\ddot{x} + \omega_0^2 x = 0$ has characteristic roots $r = \pm i\omega_0$, giving $x = A\cos(\omega_0 t + \phi)$.

Damped oscillator: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$ gives $r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$:

- Underdamped ($\gamma < \omega_0$): $x = Ae^{-\gamma t}\cos(\omega_d t + \phi)$, $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$.
- Critically damped ($\gamma = \omega_0$): $x = (C_1 + C_2 t)e^{-\gamma t}$.
- Overdamped ($\gamma > \omega_0$): $x = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ with $r_{1,2} < 0$.

Driven undamped oscillator: $\ddot{x} + \omega_0^2 x = (F_0/m)\cos(\omega_{\text{ext}} t)$.

For $\omega_{\text{ext}} \neq \omega_0$: guess $x_p = A_p \cos(\omega_{\text{ext}} t)$, giving $A_p = (F_0/m)/(\omega_0^2 - \omega_{\text{ext}}^2)$.

For $\omega_{\text{ext}} = \omega_0$ (resonance): the cosine guess fails; try $x_p = Bt\sin(\omega_0 t)$, giving $B = F_0/(2m\omega_0)$. This secular growth $x_p = \frac{F_0}{2m\omega_0} t \sin\omega_0 t$ is the hallmark of resonance: the amplitude grows linearly in time without bound. In practice, nonlinearities and damping always limit the amplitude, but this explains why driving a system near its natural frequency can produce dangerously large oscillations (e.g., the Tacoma Narrows Bridge collapse).

A.6.1 Resonance and the Quality Factor

The resonance properties of the driven damped oscillator are characterized by the **quality factor**:

$$Q = \frac{\omega_0}{2\gamma} = \frac{\omega_0 m}{b}. \tag{A.25}$$

Physically, $Q/(2\pi)$ is the number of oscillation cycles for the energy to decay to $1/e$ of its initial value. A high-Q oscillator rings for many cycles before damping out.

At resonance in the weak-damping limit ($\gamma \ll \omega_0$), the peak amplitude is:

$$A_{\text{max}} \approx \frac{F_0}{k} \cdot Q,$$

so the amplitude at resonance is Q times the static displacement F_0/k. The full width at half maximum (FWHM) of the resonance curve (measured where the amplitude falls to $A_{\text{max}}/\sqrt{2}$) is:

$$\Delta\omega \approx \frac{\omega_0}{Q} = 2\gamma.$$

A higher Q means a sharper, narrower resonance peak. Typical values: a guitar string has $Q \sim 10^3$; a quartz crystal oscillator $Q \sim 10^5$; an optical cavity $Q \sim 10^{10}$.

Driven damped oscillator (steady state): Using the complex exponential method $x_p = \text{Re}[\tilde{A}e^{i\omega_{\text{ext}}t}]$:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_{\text{ext}}^2)^2 + (2\gamma\omega_{\text{ext}})^2}}, \qquad \tan\delta = \frac{2\gamma\omega_{\text{ext}}}{\omega_0^2 - \omega_{\text{ext}}^2}.$$

A.7 Energy Methods for ODEs

For a conservative system $m\ddot{x} = F(x) = -\mathrm{d}U/\mathrm{d}x$, multiplying both sides by $\dot{x}$ and recognizing the time derivatives yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\dot{x}^2 + U(x)\right] = 0 \quad \Longrightarrow \quad E = \frac{1}{2}m\dot{x}^2 + U(x) = \text{const.}$$

This reduces the second-order ODE to a first-order separable equation:

$$\dot{x} = \pm\sqrt{\frac{2}{m}[E - U(x)]}, \qquad \mathrm{d}t = \pm\frac{\mathrm{d}x}{\sqrt{2[E - U(x)]/m}}. \tag{A.26}$$

Integrating the right side gives $t(x)$, which can (in principle) be inverted to find $x(t)$.

Example. For SHM ($U = \frac{1}{2}kx^2$, $E = \frac{1}{2}kA^2$):

$$t = \int_A^x \frac{\mathrm{d}x'}{\sqrt{(k/m)(A^2 - x'^2)}} = \frac{1}{\omega_0}\arccos\left(\frac{x}{A}\right),$$

giving $x = A\cos(\omega_0 t)$—the familiar SHM solution, recovered purely from energy conservation.

This method is particularly powerful for problems where the force depends on position but the equation of motion is not easily solvable directly, such as the Kepler problem (Chapter 16), the gravitational free-fall collision time (Problem 15.10), and effective-potential analyses (Chapter 15, Problem 15.9).

Appendix B

Multivariable Calculus

B.1 Partial Derivatives and the Gradient

For a scalar function $f(x, y, z)$, the **gradient** is:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}.$$

In mechanics: $\boldsymbol{F} = -\nabla U$ gives the force from a potential energy function.

B.2 Line Integrals

Work: $W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$. If $\boldsymbol{F} = -\nabla U$, then $W = U(A) - U(B)$ (path-independent).

B.3 Divergence, Curl, and Laplacian

$\nabla \cdot \boldsymbol{F} = \partial F_x/\partial x + \partial F_y/\partial y + \partial F_z/\partial z$.

$\nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}$. A force is conservative iff $\nabla \times \boldsymbol{F} = \boldsymbol{0}$.

$\nabla^2 f = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2$. Gravity: $\nabla^2 \Phi = 4\pi G\rho$ (Poisson's equation).

B.4 Integration in Curvilinear Coordinates

B.4.1 Polar Coordinates

The transformation from Cartesian to polar coordinates is $x = r\cos\theta$, $y = r\sin\theta$. The Jacobian of this transformation is:

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore the area element is $\mathrm{d}A = |J|\,\mathrm{d}r\,\mathrm{d}\theta = r\,\mathrm{d}r\,\mathrm{d}\theta$. Geometrically, a small patch at distance r has radial width $\mathrm{d}r$ and arc length $r\,\mathrm{d}\theta$, giving area $r\,\mathrm{d}r\,\mathrm{d}\theta$.

B.4.2 Cylindrical Coordinates

The transformation is $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$. The Jacobian is:

$$J = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho(\cos^2\phi + \sin^2\phi) = \rho.$$

The volume element is $\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$. The three unit vectors are obtained from the columns of the Jacobian matrix, normalized to unit length:

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &= \frac{\partial \boldsymbol{r}/\partial\rho}{|\partial \boldsymbol{r}/\partial\rho|} = \cos\phi\,\hat{\boldsymbol{\imath}} + \sin\phi\,\hat{\boldsymbol{\jmath}}, \\
\hat{\boldsymbol{\phi}} &= \frac{\partial \boldsymbol{r}/\partial\phi}{|\partial \boldsymbol{r}/\partial\phi|} = -\sin\phi\,\hat{\boldsymbol{\imath}} + \cos\phi\,\hat{\boldsymbol{\jmath}}, \\
\hat{\boldsymbol{z}} &= \hat{\boldsymbol{k}}.
\end{aligned}$$

One can verify that $\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\phi}} = 0$, $\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}}$ (right-handed), and all have unit length.

B.4.3 Spherical Coordinates

The transformation is $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. The Jacobian is:

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}.$$

Expanding along the bottom row:

$$\begin{aligned}
J &= \cos\theta\big(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\big) \\
&\quad + r\sin\theta\big(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\big) \\
&= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta = r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta.
\end{aligned}$$

Therefore the volume element is $\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$. On a sphere of constant radius R, the surface area element is $\mathrm{d}A = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$, and integrating over the full sphere:

$$A = \int_0^{2\pi}\int_0^{\pi} R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = R^2(2\pi)(2) = 4\pi R^2.$$

The unit vectors are derived from the unnormalized tangent vectors $\partial\boldsymbol{r}/\partial r$, $\partial\boldsymbol{r}/\partial\theta$, $\partial\boldsymbol{r}/\partial\phi$:

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \frac{\partial\boldsymbol{r}}{\partial r}\Big/\left|\frac{\partial\boldsymbol{r}}{\partial r}\right| = \sin\theta\cos\phi\,\hat{\boldsymbol{\imath}} + \sin\theta\sin\phi\,\hat{\boldsymbol{\jmath}} + \cos\theta\,\hat{\boldsymbol{k}}, \\
\hat{\boldsymbol{\theta}} &= \frac{1}{r}\frac{\partial\boldsymbol{r}}{\partial\theta}\Big/\left|\frac{1}{r}\frac{\partial\boldsymbol{r}}{\partial\theta}\right| = \cos\theta\cos\phi\,\hat{\boldsymbol{\imath}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\jmath}} - \sin\theta\,\hat{\boldsymbol{k}}, \\
\hat{\boldsymbol{\phi}} &= \frac{1}{r\sin\theta}\frac{\partial\boldsymbol{r}}{\partial\phi}\Big/\left|\frac{1}{r\sin\theta}\frac{\partial\boldsymbol{r}}{\partial\phi}\right| = -\sin\phi\,\hat{\boldsymbol{\imath}} + \cos\phi\,\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

These form a right-handed orthonormal basis: $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$, and all dot products between distinct unit vectors vanish.

B.4.4 The Gradient in Curvilinear Coordinates

In cylindrical coordinates: $\nabla f = \frac{\partial f}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$.

In spherical coordinates: $\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}$.

The factors of $1/\rho$, $1/r$, and $1/(r\sin\theta)$ arise from the scale factors (the norms of the unnormalized tangent vectors). The Laplacian in spherical coordinates is:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.$$

This is used in Chapter 15 (Gravitation) for Poisson's equation and in quantum mechanics for the hydrogen atom.

B.5 Derivation of Polar Kinematics

From $\hat{\boldsymbol{r}} = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$:

$$\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = \dot\theta\,\hat{\boldsymbol{\theta}}, \qquad \frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\dot\theta\,\hat{\boldsymbol{r}}.$$

Then $\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{\boldsymbol{r}}) = \dot r\hat{\boldsymbol{r}} + r\dot\theta\hat{\boldsymbol{\theta}}$ and differentiating again: $\boldsymbol{a} = (\ddot r - r\dot\theta^2)\hat{\boldsymbol{r}} + (r\ddot\theta + 2\dot r\dot\theta)\hat{\boldsymbol{\theta}}$.

Appendix C

Linear Algebra for Mechanics

C.1 Rotation Matrices

A rotation by angle θ counterclockwise in the xy-plane transforms coordinates as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R(\theta) \begin{pmatrix} x \\ y \end{pmatrix}, \qquad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Key properties: R is orthogonal ($R^T R = I$, $\det R = 1$), $R(\alpha)R(\beta) = R(\alpha+\beta)$ (rotations compose by adding angles), and $R^{-1}(\theta) = R(-\theta) = R^T(\theta)$.

This is precisely how the polar unit vectors relate to the Cartesian ones: $\hat{\boldsymbol{r}} = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$, which is $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}})^T = R(\theta)(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}})^T$.

In three dimensions, rotations about the coordinate axes are:

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

An arbitrary 3D rotation can be decomposed into a product of these (Euler angles), though we do not pursue this further.

C.2 The Moment of Inertia Tensor

For rotation about an arbitrary axis, the angular momentum is not generally parallel to the angular velocity. The full relationship is:

$$\boldsymbol{L} = \mathbf{I}\,\boldsymbol{\omega}, \tag{C.1}$$

where $\mathbf{I}$ is the 3×3 symmetric **moment of inertia tensor**:

$$\mathbf{I} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix}.$$

The diagonal elements are the moments of inertia about the coordinate axes: $I_{xx} = \sum m_i(y_i^2 + z_i^2)$, etc. The off-diagonal elements are the **products of inertia**: $I_{xy} = \sum m_i x_i y_i$, etc.

When $\boldsymbol{\omega}$ is along one of the coordinate axes (say $\hat{\boldsymbol{k}}$), $\boldsymbol{L} = -I_{xz}\omega\,\hat{\boldsymbol{\imath}} - I_{yz}\omega\,\hat{\boldsymbol{\jmath}} + I_{zz}\omega\,\hat{\boldsymbol{k}}$—which is *not* parallel to $\boldsymbol{\omega}$ unless $I_{xz} = I_{yz} = 0$. This is why unbalanced wheels wobble: $\boldsymbol{L}$ precesses around $\boldsymbol{\omega}$, requiring a torque from the bearings.

Principal axes. Since $\mathbf{I}$ is real and symmetric, it has three real eigenvalues I_1, I_2, I_3 (the **principal moments of inertia**) and three mutually perpendicular eigenvectors (the **principal axes**). When $\boldsymbol{\omega}$ aligns with a principal axis, $\boldsymbol{L} = I_k\boldsymbol{\omega}$ is parallel to $\boldsymbol{\omega}$, and the scalar relation $L = I\omega$ that we use throughout Chapters 11–13 applies. For symmetric objects (spheres, cylinders, rectangular boxes aligned with coordinate axes), the coordinate axes are already principal axes.

C.3 Eigenvalues and the Characteristic Equation

The eigenvalues of a matrix $\mathbf{A}$ are the values λ for which $\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}$ has nontrivial solutions. They satisfy the **characteristic equation**:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{C.2}$$

For a 2×2 matrix, this is a quadratic; for 3×3, a cubic.

In this textbook, eigenvalues appear in two contexts:

- **Moment of inertia tensor**: the eigenvalues are the principal moments I_1, I_2, I_3.
- **Linear ODEs**: the equation $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$ has the characteristic equation $r^2 + 2\gamma r + \omega_0^2 = 0$. Its roots $r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$ determine whether the system is underdamped (complex roots), critically damped (repeated root), or overdamped (two distinct real roots). See Appendix A.6.

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